

NEUVIEME COLLOQUE SUR LE TRAITEMENT DU SIGNAL ET SES APPLICATIONS

407



NICE du 16 au 20 MAI 1983

SOFT DECODING USING A TRELLIS FOR A CONCATENATED SYSTEM DECODAGE PONDERE PAR TREILLIS D'UN SYSTEME CONCATENE

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RESUME

Le codage concaténé a été introduit par Forney comme un moyen pratique de réaliser de longs codes et donc d'obtenir des probabilités d'erreur faibles. Les systèmes les plus usuels emploient deux étages: l'extérieur est souvent un code de Reed-Solomon tandis que l'intérieur est un code soit en blocs, soit convolutif. Nous considérons ici pour code intérieur un code en blocs court qu'il est possible de décoder à vraisemblance maximale. Nous supposons de plus que ce décodeur fournit le vecteur des probabilités correspondant à chaque symbole du mot de Reed-Solomon reçu. La technique de décodage (à décision souple) proposée ici pour le code extérieur emploie un treillis, c'est-à-dire un ensemble de chemins qui joignent deux points; chacun d'eux est constitué de N branches successives, chacune partant d'un noeud et aboutissant à un noeud. En général, chaque noeud est relié à d'autres (qui le précèdent ou le suivent) par q branches (taille de l'alphabet). Un élément j de $GF(q)$ est associé à chaque branche, de telle sorte que chacun des chemins dans le treillis représente un mot du code. De plus, une grandeur réelle qui mesure la probabilité que le symbole émis correspondant égale j , conditionnellement au signal reçu, est supposée disponible pour chaque branche. Le décodage peut en principe s'effectuer par l'algorithme de Viterbi, mais c'est pratiquement impossible à cause du volume de calcul nécessaire. Nous essayons ici de réduire la complexité du décodage en n'employant qu'un treillis partiel. Nous montrons d'abord que le décodage optimal n'exige pas de considérer la totalité du treillis. La plus grande simplification est obtenue si l'on commence par réordonner les symboles reçus en fonction de leur fiabilité. Une simplification plus importante (au détriment de l'optimalité) s'obtient en n'utilisant qu'un sous-ensemble de l'alphabet pour les symboles les plus fiables. Ce sous-ensemble est obtenu en comparant les probabilités avec une constante dont le choix détermine la probabilité d'erreur finale. Un exemple a été étudié par simulation pour le code de Reed-Solomon (7,5) sur $GF(8)$ et le code à longueur maximale (7,3). Des suggestions sont faites quant au choix de la constante.

SUMMARY

Concatenated coding was introduced by Forney as a practical technique for implementing long codes, thereby achieving low error probabilities. The most common systems use two coding stages: the outer one is often a Reed-Solomon code, while the inner one can be either a block or a convolutional code. Here we assume that the inner code is a short block code for which a maximum-likelihood decoder is available. Moreover, we assume that the output of this decoder is in the form of a probability vector, corresponding to each symbol of the received Reed-Solomon codeword. The proposed soft-decision decoding procedure for the outer code uses a trellis i.e., a set of paths connecting two points; each path is made of N successive branches, each of which stems from, and terminates in, a node. Each node in general is connected to preceding or successive ones by q branches (the alphabet size). An element j of $GF(q)$ is associated with each branch, so that each path in the trellis represents a codeword. Moreover, a real value which measures the probability that the corresponding transmitted symbol equals j , conditioned on the received signal, is assumed to be available for each branch. Decoding may then be performed by the Viterbi algorithm, but this is impractical due to the large amount of computation required. We try here to reduce the decoding complexity by using a partial trellis. We first show that optimum decoding does not require considering the full trellis. The largest simplification results if the received symbols are first sorted according to their reliability. Further simplification (at the expense of optimality) results in using only a subset of the alphabet for the most reliable symbols. This subset is obtained by comparing the probabilities with a constant, whose choice determines the final decoding error probability. A simple example has been studied by simulation using the Reed-Solomon code (7,5) over $GF(8)$ and a maximal length code (7,3). Suggestions for choosing the constant are presented.



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I. CONCATENATED SYSTEM

Concatenation, as introduced by FORNEY /1/, is a means for combining two or more codes using their direct product over different fields. The general structure of a concatenated system is shown in Fig.1 where a sequence of kK binary information symbols is partitioned into K k -tuple subblocks which are considered as elements of $GF(q)$, $q=2^k$. These K subblocks are then encoded by an (N,K) block code, linear over $GF(q)$, called the outer code (most commonly a Reed-Solomon code). The resulting codeword is denoted by $c = [c_1, c_2, \dots, c_N]$. Each element c_i written as a k -symbol binary vector is encoded by a linear (n,k) binary code, called the inner code. A binary code of length nN , dimension kK and rate $(k/n) \cdot (K/N)$ results whose Hamming distance is greater than or equal to the product of the Hamming distances of the used codes /2/.

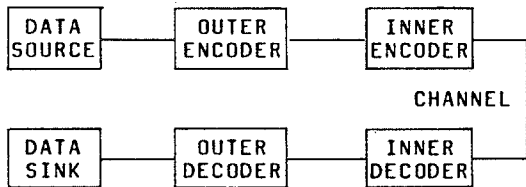


Fig.1 Block diagram of a concatenated system

Assuming the channel errors to be random, the errors made by the inner decoder will appear as bursts, most of which will be corrected by the outer decoder. Very low decoding error probability will therefore result. We assume that maximum likelihood decoding (MLD) is used for the inner code, and that its decoder will produce a reliability indication for the received symbols. As proposed here, the outer RS decoder is a simplified maximum likelihood decoder which makes use of the reliability information thus provided.

If we assume that the received binary words are independent, then the RS symbols will also be independent. If the RS symbol error probability at the decoder input, say P_e , can be determined, by simulation for example, then for an RS code of length $N = 2^k - 1$ and error correcting capability t , $t = \lfloor (N-k)/2 \rfloor$, a symbol error will occur if the number of symbol errors exceeds t , so the RS decoder error probability can be bounded by the binomial expressions :

$$(1) \quad P_w \leq \sum_{i=t+1}^N \binom{N}{i} P_e^i (1 - P_e)^{N-i}$$

$$(2) \quad P_s \leq \sum_{i=t+1}^N \left(\frac{i}{N}\right) \binom{N}{i} P_e^i (1 - P_e)^{N-i}$$

where P_w and P_s are the word and symbol error probabilities at the RS decoder output respectively. The factor (i/n) in (2) is the average number of errors per symbol so (2) can be considered as an estimate of the symbol error probability. In fact if the

number of erroneous symbols exceeds t , the RS decoder would cause more symbol errors, t errors at most, so the symbol error probability can be bounded by /3/,

$$P_s(e) \leq \sum_{i=t+1}^N \binom{N}{i} P_e^i (1 - P_e)^{N-i}$$

$$(3) \quad \leq (1 + t/N) P_s$$

The overall error probability of a concatenated system can be estimated if the number of bit errors in each erroneous RS symbol is constant. This is the case when the inner code is a constant weight code, i.e., simplex or maximal length code described as a $(2^k-1, k, 2^{k-1})$ code. Thus the bit error probability at the output of the concatenated system is upper bounded by

$$(4) \quad P_b \leq \frac{2^{k-1}}{2^k-1} (1 + t/N) P_s$$

II. TRELLIS DECODING, DEFINITION AND NOTATION

The idea of using a trellis for decoding block codes is not new /4, 5/. It may be used to represent a linear code or its cosets /6/. In order to define it, we found helpful to use a monomial representation besides the vectorial one, for example if u is an L -dimensional vector with elements from $GF(q)$, its monomial representation in terms of L indeterminates $X = \{x_1, x_2, \dots, x_L\}$ is

$$(5) \quad U(X) = x_1^{u_1} x_2^{u_2} \dots x_L^{u_L}$$

Consider a linear code $C(n,k)$ over $GF(q)$ having the parity check matrix

$$(6) \quad H = [h_1, h_2, \dots, h_n]$$

where h_i , $i = 1, 2, \dots, n$, is a column vector of dimension $n-k$. Let $T(X,Y)$ be the polynomial associated with the matrix H and defined as

$$(7) \quad T(X,Y) = \prod_{i=1}^n \left[\sum_{j=0}^{q-1} (y_i H_i(X))^j \right]$$

where $H_i(X)$ is the monomial representation of h_i in terms of $X = \{x_1, x_2, \dots, x_{n-k}\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ is another set of n indeterminates. The exponents of the indeterminates are computed in $GF(q)$. $T(\dots)$ can be expanded with the help of a trellis diagram, where the exponents of the indeterminates X are represented by a vector \underline{s} which we call a state. The states, written as $n-k$ q -ary digits, will be ordered according to their values, say s , from 0 to $q^{n-k}-1$. The corresponding state will be denoted by $\underline{s}(s)$ and s will be plotted in ordinate. The abscissa will be an integer i , $i = 0, 1, \dots, n$ to be referred to as the level. The trellis from the zero level to the i th one represents the product of the first i factors in the expansion of (7). A level i and a state \underline{s} will define a node. Each node can be

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connected to the nodes in the preceding or the next level by at most q branches labeled by the corresponding indeterminates y_i^j , $j = 0, 1, \dots, q-1$. At the zero level there is a single state $S=0$. The exponent of the indeterminates y_i of the monomial associated with a certain path will represent a codeword. Since each codeword \underline{c} in C must satisfy the code constraints

$$\underline{c} \cdot H^T = 0$$

then all the paths in the trellis have to converge to the zero state at the level n. For a systematic code C(n,k), the trellis is in general made of three parts: in the first part all the branches diverge from the zero state at the zero level, in the central part divergence as well as convergence occur, while in the last part or tail the branches converge toward the zero state at the level n. For high rate codes, i.e., those having $k \geq n-k$, all the states can be reached in the central part but this is not the case for low rate ones.

III. CHANNEL MODELLING AND WEIGHTING OF THE INNER DECODER OUTPUT

Let p_{ij} be the probability that the ith transmitted symbol equals j conditioned on the received signal, assuming that the received symbols are independent. The probability p_{ij} is not directly known, but according to Bayes' rule it can be expressed in terms of the transition probabilities of the channel and the transmission probabilities. We assume here equiprobable transmitted symbols. We call the vector \underline{p}_i whose components are p_{ij} , $j = 0, 1, \dots, q-1$, such that

$$\sum_{j=0}^{q-1} p_{ij} = 1,$$

the a priori probability vector of the ith transmitted symbol. Substituting p_{ij} for y_i^j in the monomial associated with a certain path (or codeword) \underline{c} whose branches (or symbols) equal the exponents of y_i for $i=1, \dots, n$, we can get the a posteriori probability that the word \underline{c} is transmitted simply by dividing this monomial by the sum of the monomials associated with all paths in the trellis. This can be expressed as

$$(8) \quad \Pr(\underline{c}) = \prod_{\substack{i=1 \\ \underline{c} \in C}}^n p_{i c_i} / \left[\sum_{\underline{c} \in C} \prod_{i=1}^n p_{i c_i} \right]$$

In maximum likelihood decoding, one looks for the codeword \underline{c} having the largest probability $\Pr(\underline{c})$. In our case where each codeword of the inner code C(n,k) corresponds to a symbol of the outer code C(N,K), we keep all the a posteriori probabilities of these codewords, to be used as the a priori probability vector for the corresponding symbol in the outer code. In the binary case we can divide both the numerator and the denominator in (8) by the a posteriori probability of the zero word, which results in

$$(9) \quad \Pr(\underline{c}) = \prod_{\substack{i=1 \\ \underline{c} \in C}}^n (p_i/q_i)^{c_i} / \left[\sum_{\underline{c} \in C} \prod_{i=1}^n (p_i/q_i)^{c_i} \right]$$

where $p_i = 1 - q_i$ is the probability that the ith symbol equals 1 conditioned on the received signal. Letting $a_i = \log(q_i/p_i)$ be the algebraic value of the ith received symbol, we can write (9) in the following form

$$(10) \quad \Pr(\underline{c}) = \exp\left(-\sum_{i=1}^n c_i a_i\right) / \left[\sum_{\underline{c} \in C} \exp\left(-\sum_{i=1}^n c_i a_i\right) \right]$$

a_i can be thought of as available at the output of a matched filter, assuming that the noise is additive. Here we will consider a simple inner code so we can calculate the above probability with the help of a table containing all the codewords. In this way we can get the a priori probability vector \underline{p}_i of $q=2^k$ components associated with the corresponding symbol in the outer code.

IV. WEIGHTING OF THE OUTER DECODER

The choice of an RS code as outer code can provide a simple means for using the probability vectors, as available from the inner decoder, since it is a maximum distance separable code (MDS), i.e., whose minimum distance is $D=N-K+1$. With such codes, a codeword can be found if K symbols are known regardless of their locations. We call such a set of K symbols an information set. Assume that with each received symbol, c_i , of the outer code we associate a single element j from $GF(q)$ such that p_{ij} is the largest component in \underline{p}_i . From the N elements associated with the received symbols we choose the most probable K ones, i.e., those corresponding to the largest p_{ij} . From this information set we can find the remaining N-K symbols either by solving a set of linear equations in the N-K unknown symbols, or by the use of a table containing all the codewords if possible. If there is no error in the information set, the transmitted codeword can be found. Thus the decoder in this case erases systematically the least reliable N-K symbols. We refer to this decoding method as best-K decoding and to the codeword found as the first word. We can expect that the performance of this method is far from the optimum, but we try to make use of such an idea together with trellis decoding to reduce the decoding complexity. As mentioned before, ... optimum decoding can be carried out with the help of a trellis whose complexity is measured, in general, by the number of states, i.e., q^{n-k} . Fig.2 shows the trellis for the extended RS code (4,2,3) over $GF(4)$. We can imagine how the trellis would look like for a moderate length code like RS (15,11,5) over $GF(16)$ where there are 65,536 states. We found that it is not necessary to consider the full trellis during the decoding process even for optimum decoding, it suffices to consider only a partial trellis.



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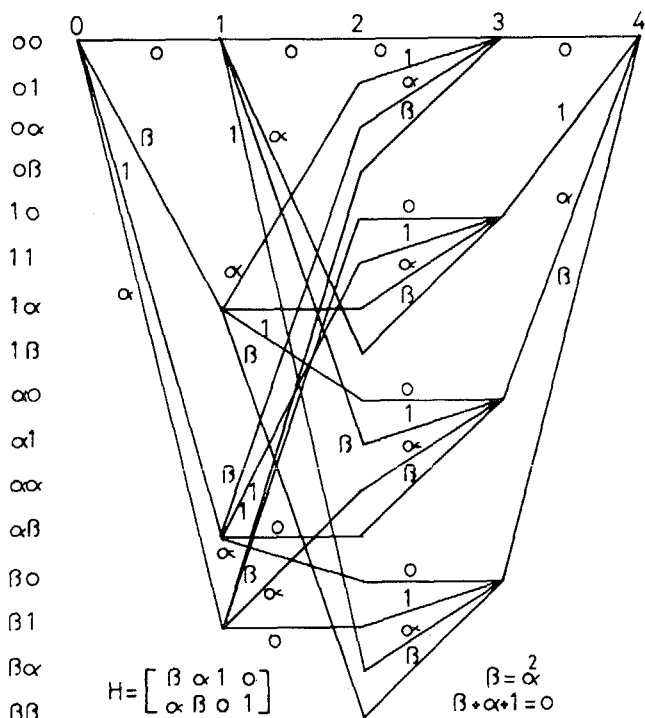


FIG.2 Trellis diagram corresponding to the code RS (4,2) over GF(4)

V. DECODING PROCEDURE

The order of the factors when expanding (7) is arbitrary so that we may have $N!$ equivalent trellises corresponding to all possible symbol permutations. Using the a priori vectors available from the inner decoding, the symbols of the outer code can be sorted according to their probability in a way which may simplify the decoding process.

CASE A

First we will use the best-K decoding idea to reduce the trellis structure as follows

(i) The components of the a priori vector $P_i = [P_{ij}]$, $j = 0, 1, \dots, q-1$ associated with the i th symbol c_i of the outer code have to be reordered by decreasing probabilities, say $P_{a_{i1}}, P_{a_{i2}}, \dots, P_{a_{iq}}$, $a_{i1}, a_{i2}, \dots \in GF(q)$.

(ii) The symbols c_i of the received word are then sorted according to the largest $P_{a_{i1}}$ for $i = 1, 2, \dots, N$, i.e., $i > j \iff P_{a_{i1}} > P_{a_{j1}}$

Let I' denote the new order of symbols.

(iii) The columns of the parity check matrix H of the outer code are permuted according to the new order of symbols.

(iv) The most reliable K symbols $a_{11}, a_{21}, \dots, a_{K1}$ form an information set, so we

can trace the path in the trellis having its first branches equal to the a_{i1} , $i = 1, 2, \dots, K$.

(v) The a posteriori probability associated with the first path is used as a threshold. In tracing the different paths in the trellis we eliminate those having a product of a priori probabilities less than the threshold. Moreover at any node value where several paths converge we consider only the path having the largest probabilities' product. The real number associated with each node will therefore be the largest product of the a priori probabilities of the path branches ending at that node, i.e.,

$$(11) \quad Z_i(j) = \max_{\beta \in GF(q)} [Z_{i-1}(m) P_{a_{i\beta}}]$$

such that

$$S_i(j) = S_{i-1}(m) + \beta h_i,$$

$$j, m \in \{0, 1, \dots, q^{N-K} - 1\}$$

where $Z_i(j)$ is the real number associated with the node j in the i th level, given the initial values $Z_0(0) = 1$ and $Z_0(1) = 0$ for $l = 1, 2, \dots, q^{N-K} - 1$. $P_{a_{i\beta}}$ is the probability that the i th symbol a_i equals β .

We have chosen a concatenated code (49,15) composed of the maximal length code (7,3,4) and the RS (7,5,3) over GF(8) to study by simulation the performance of the decoding process just described over an additive white Gaussian noise channel. The symbol error rate performance thus obtained is shown in the Fig.3.

CASE B

The first two steps are the same as (i) and (ii) in Case A.

(iii) For a given constant f , $0 \ll f \ll 1$, let m_i be the smallest integer such that

$$(12) \quad \sum_{j=1}^{m_i} P_{a_{ij}} \geq 1 - f$$

m_i is the number of the most probable elements of $GF(q)$ associated with the i th symbol.

(iv) The symbols are reordered again according to descending m_i . Let I'' represent the new order.

(v) The columns of H are permuted accordingly and we constitute the corresponding trellis after the expression (7) whose right hand side can be written as

$$(13) \quad \prod_{i=1}^K \left[\sum_{u \in U_i} P_{a_{iu}} H_i(x) \right] \cdot \prod_{i=K+1}^N \left[\sum_{j=1}^q P_{a_{ij}} H_i^j(x) \right]$$

where $U_i = \{u_{ij}\}$, $j = 1, 2, \dots, m_i$, is the subset of m_i elements from $GF(q)$ associated with the i th symbol.

The node value will be calculated according

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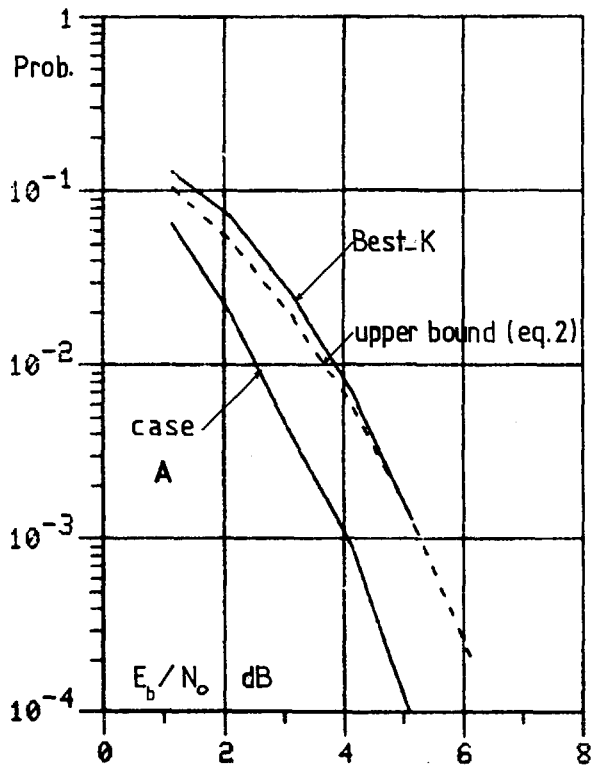


Fig.3 Symbol error rate of the outer RS(7,5) code over GF(8) for an AWGN channel (Case A)

to (11), so the number of paths in the trellis will be bounded by

$$M = m_1 m_2 \dots m_K \leq q^{N-K} = 2^{k(N-K)}$$

since all the paths will converge, after the K th level, towards the zero state at the level N . The condition (12) means that for a sufficiently small f , the path to be chosen by maximum likelihood decoding would be one of the M considered ones. Of course $f=0$ corresponds to ML decoding while $f=1$ results in the best-K decoding. Fig.4 shows the symbol error rate performance in this case for different values of f , as resulting from simulation using the same code as before.

VI. SUGGESTIONS FOR THE CHOICE OF THE CONSTANT f

The choice of the constant f is an important factor in the above procedure since the smaller the value of f is, the larger the number of considered paths. A possible choice of f is $1/q$, but for the moderate length code chosen here the use of $f \geq .05$ will lead to a non-negligible performance degradation. It is possible to find a choice criterion which guarantees that the decoding performance remains close to the optimum.

Assume that the elements of $GF(q)$ associated with each symbol of the outer code as well as the symbols of the received outer word are sorted in decreasing order according to their probabilities, then we have

$$(14) \quad \prod_{j=1}^K P_{a_{j1}} \cdot \prod_{j=K+1}^N P_{a_{jq}}$$

as a lower bound of the probabilities' product associated with the first path. The first part corresponds to the best K symbols and the second one corresponds to the least reliable $N-K$ symbols. Since the code is an MDS one, there is a single codeword that contains the best K symbols. On the other hand, the a priori probabilities of the last $N-K$ symbols will be greater than or equal to the second part. So any path which will have a smaller probability can be eliminated just after observing the first branches without any risk of eliminating the correct one if f_1

$$(15) \quad f_1 \prod_{j=2}^N P_{a_{j1}} = \prod_{j=1}^K P_{a_{j1}} \prod_{j=K+1}^N P_{a_{jq}}$$

$$\text{or} \quad f_1 = P_{a_{11}} \prod_{j=K+1}^N (P_{a_{jq}} / P_{a_{j1}})$$

where the left hand term in (15) is an estimate of the probabilities' product associated with the path eliminated just after its first branch, while the right hand side expression is the bound of the product associated with the first path. So we can choose for the i th symbol the constant

$$(16) \quad f_i = P_{a_{i1}} \prod_{j=K+1}^N (P_{a_{jq}} / P_{a_{j1}})$$

The ratio $P_{a_{jq}} / P_{a_{j1}}$ is a measure of the deviation between the two extremes. If this ratio is too small, we will have a small f_i which may not lead to a significant simplification. The choice of the constant

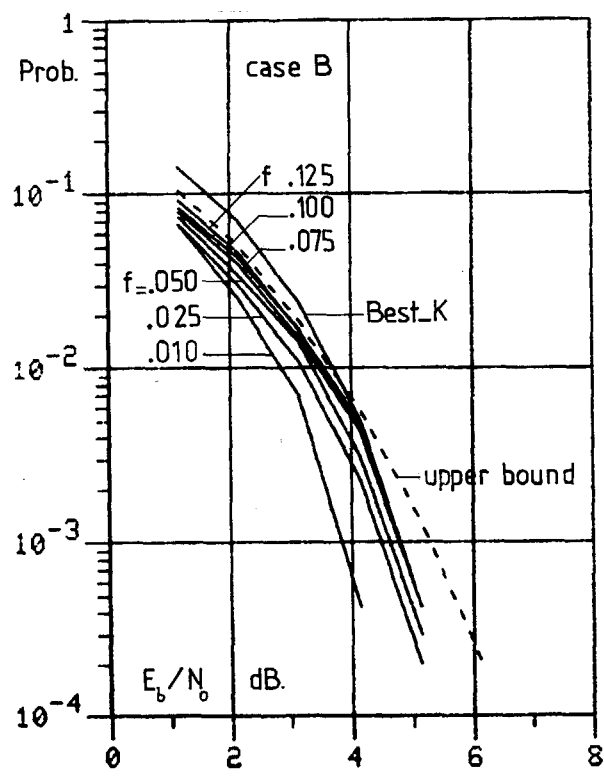


Fig.4 Symbol error rate of the outer RS(7,5) code over an AWGN channel for different values of f . (Case B)



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f_i according to (16) is only necessary to guarantee the optimality. In order to have a real simplification the value of f_i must be much greater than that determined by (16). It may be thought of replacing the denominator in (16) by the product of probabilities associated with the last $N-K$ symbols in the first path, so we will have

$$f_i = P_{a_{i1}} \prod_{\substack{j=K+1 \\ c \in C}}^N (P_{a_{jq}} / P_{a_{jc}})$$

where c is the path having its first branches equal to the best K symbols. At high SNR P_a becomes nearly zero, so we will get very small f_i , then a large number of paths will be considered. We found by simulation for the code RS (7,5) that the decoding performance with f_i determined by (16) is close to that obtained by a constant value $\ll .001$ but with a larger number of paths.

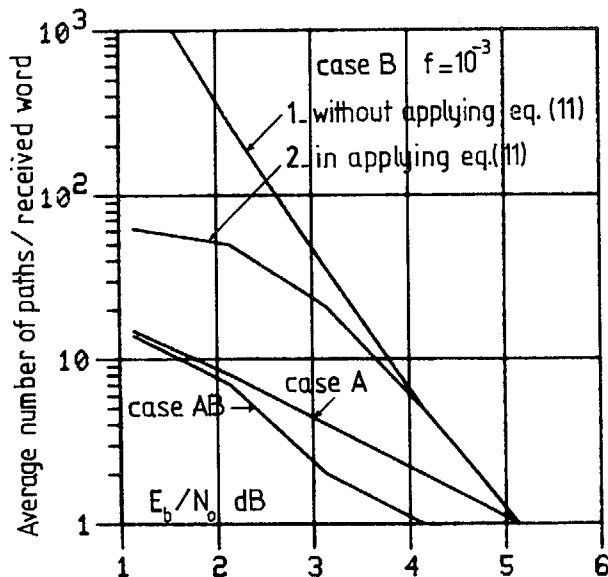


Fig.5 Average number of paths considered in the trellis in the different cases to get the same performance

If we use the threshold determined by the first path as well as a constant f we can get a further reduction in the number of paths considered in the trellis. We refer to this case as AB in Fig.5 showing the average number of paths required to obtain the same performance in the different cases. The average number of paths needed if the rule (11) is not applied is also shown in the same figure.

VII. REMARKS

Several works have been devoted to study the performance of concatenated systems. The main interest there was to find the inner code or decoding method which could improve the overall performance of the system together with the use of conventional RS decoders. The inner codes are not necessarily block codes e.g., the concatenated system used for deep space communications where a convolutional inner code is

used. Here on the contrary we tried to improve the system performance by using a soft decoding technique for the outer code. The decoding method proposed here depends upon the use of a decoding trellis as that used for maximum likelihood decoding. In fact we use a partial trellis where a part of the nodes and the branches are not considered. This resembles (in case B) the method given in /8/ but here we consider the q -ary case. For the example of the code RS (7,5,3) over GF(8) treated here, the path number is 32,768 in the full trellis. This number is reduced to 64, i.e., the number of states, if we consider only the most probable paths, and with the decoding procedure proposed here this number can be reduced to 16 at low signal to noise ratios. With the decoding algorithm proposed in /7/ this number can be reduced to only 5 paths. The decoding gain at symbol error rate 10^{-3} is about 3 dB for the moderate length code used here.

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