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FOKKER-PLANCK STUDY OF THE PHASE-LOCKED-LOOP: BEHAVIOR ON THE REAL LINE

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RESUME

La plupart des publications qui concernent le boucle à verrouillage de phase (PLL) étude le comportement de l´erreur de phase cyclique. La solution en régime permanent de l´équation de Fokker-Planck (F/P) pour ce problème est connue. Une approche usuelle étend cet étude à la droite réelle en representant le processus d´erreur par l´addition d´un processus cyclique et un processus de comptage. On ne trouve pas dans la littérature des études permettant conclure de la validité de cette decomposition. Ici, on l´essaie en comparant la densité congruente resultant avec la solution directe de l´équation de F/P sur la droite. Aussi, on analyse l´évolution de l´enveloppant de la densité a vis d´obtenir information sur le processus de comptage.

Ce travail present l´integration numérique directe de l´équation de F/P sur la droite réelle. Contrairement à la solution cyclique le domaine de la fonction densité de probabilité du processus d´erreur n´est pas bornée. En plus, parce que la solution permanente est nulle partout, le problème devient un probleme transitoire. Cet exposé present un algorithme numérique d´integration sur la droite. L´algorithme exploite la structure du problème pour reduire le volume de calcul. Le nombre de points sur la droite est augmenté adaptativement autant que l´horizon temporel augmente. Une analyse de convergence compare des méthodes numériques alternatives se basant sur trois restrictions qui établient des rapports entre: i) le pas temporel et les paramètres statistiques (les coefficients de l´équation aux derivées partielles); ii) le pas spatial et les paramètres statistiques; iii) les deux pas. Dans le cas etudié, on verifie que pour la méthode explicite et l´algorithme de Crank-Nicholson la troisième condition est la plus restrictive, étant trivialement satisfaite pour la méthode totalement implicite. On a conclu que pour certains valeurs des paramètres la méthode de Crank-Nicholson conduit a une solution non positive, tandis que la méthode totalement implicite n´offre pas cette difficulté, étant celle qu´on a mis en oeuvre.

SUMMARY

The work usually reported on the literature on the Phase Locked Loop (PLL) foccus attention on the cyclic phase error behavior. For this problem, the steady state solution of the associated Fokker-Planck (F/P) equation is known. The common approach to extend to the real line the study of the PLL error process is to decompose it as the sum of a cyclicized component and a counting process. No concluding evaluation of the validity of this decomposition exists in the litterature. Herein, it is assessed by comparing the resultant congruent density with the direct line solution of the F/P equation. This entails the study of the F/P equation on the real line. Also, to obtain a feeling for the statistics of the counting process, the evolution of the line density envelope has been analysed.

The present work carries out the direct numerical integration of the F/P equation on the real line. Contrasting with the cyclic solution, the support of the probability density function of the error process is now unbounded (the entire real line). Also, the steady state solution being the trivial one, the line problem becomes intrinsecally a transient problem. The paper describes an algorithm for the numerical line integration. The algorithm explores the structure of the problem to reduce its computational burden. It adaptively increases the spatial grid as the time horizon increases. A convergence analysis compares alternative numerical methods, based on three restrictions which relate i) the temporal increment with the statistical parameters (coefficients of the pde); ii) the spatial increment with the statistical parameters; iii) the temporal and spatial increments. For the case under study, it turns out that for the explicit method and the Crank- Nicholson algorithm the third condition is more restrictive than the other two, while being trivially satisfied for the completely implicit method. It is concluded that for certain parameter values, the Crank-Nicholson method incurs into problems of nonpositiveness of the solution. It is also shown that the completely implicit method does not experience these problems, being the one implemented. The numerical study is also used to assess the decomposition.

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I - INTRODUCTION

The phase process assumed here is described by the stochastic differential equation

$$dx(t)=dB(t) \qquad x(0)=0 \tag{1}$$

where B(t) is a zero mean Brownian motion with variance parameter q.

The observation model is

$$dy(t)=\begin{bmatrix}\sin(x(t))\\ \cos(x(t))\end{bmatrix} dt+\begin{bmatrix}dn_1(t)\\ dn_2(t)\end{bmatrix} \tag{2}$$

where n(t) is a zero mean Brownian vector process independent of B(t) with covariance matrix $rI_2\, t$[1].

The estimate given by the PLL satisfies

$$d\hat{x}(t)=K[\sin(x(t)-\hat{x}(t))-dw(t)] \tag{3}$$

where

$$dw(t)=dn_2(t)\cos(\hat{x}(t))-dn_1(t)\sin(\hat{x}(t)) \tag{4}$$

and

$$k=\sqrt{q/r} \tag{5}$$

The phase error

$$e(t)=x(t)-\hat{x}(t) \tag{6}$$

is given by

$$de(t)=-K\sin(e(t))dt-Kdw(t)+dx(t) \tag{7}$$

Following [1], it can be shown that w(t) is correctly modeled, in the bandwith of interest, by a Brownian motion with variance rt. The phase error is a Markov process and its transition probability density function (tpdf), $p(e,t;e_0,t_0)$[2], satisfies the Fokker-Planck equation

$$\partial p/\partial t=\partial(pK\sin e)/\partial e+(1/2)(K^2r+q)\,\partial^2p/\partial e^2 \tag{8}$$

subject to the boundary conditions

$$p(\pm\infty,t)=0 \tag{9}$$

and the initial condition

$$p(e,t)=\delta(e) \tag{10}$$

(1) I_2 stands for a 2x2 identity matrix

(2) For simplicity we use $p=p(e,t)=p(e,t;e_0,t_0)$

In some problems, we are only concerned with the so called cyclic phase error defined by

$$\tilde{e}(t)=(e(t))\bmod 2\pi \tag{11}$$

with density

$$\tilde{p}(\tilde{e},t)=\sum_{k=-\infty}^{+\infty} p(\tilde{e}+2k\pi,t) \quad ,-\pi<\tilde{e}<\pi \tag{12}$$

which satisfies Eq.(8), with (9) replaced by

$$\tilde{p}(-\pi,t)=\tilde{p}(\pi,t) \qquad \partial\tilde{p}/\partial\tilde{e}\Big|_{\tilde{e}=-\pi}=\partial\tilde{p}/\partial\tilde{e}\Big|_{\tilde{e}=\pi} \tag{9a}$$

This equation has a steady state solution given by (see [1])

$$\tilde{p}(\tilde{e})=\exp(\alpha\cos\tilde{e})/2\pi I_0(\alpha) \tag{13}$$

where

$$\alpha=1/\sqrt{qr} \tag{14}$$

is the signal to noise ratio in the bandwith of the loop.

It is easy to show that the density of the line error can be factored according to

$$p(e,t)=E(e,t)\,[\tilde{p}(\tilde{e})*\textstyle\sum\delta(e-2k\pi)] \tag{15}$$

where * stands for convolution. The envelope E(e,t) satisfies (see [2]) the Backward Kolmogorov equation associated with problem (7), i.e.,

$$\partial E/\partial t=-K\sin e\,\partial E/\partial e+(1/2)(K^2r+q)\,\partial^2E/\partial e^2 \tag{16}$$

In this paper, an algorithm is presented to solve Eqs.(8) and (16) together with a convergence study. The properties of the envelope will be used to provide an interpretation to the phase error process.

II - CONVERGENCE ANALYSIS

Eqs. (8) and (16) are parabolic partial differential equations. Here, a finite difference method is applied to solve them numerically.

As the coefficients of the equations are not constants, there are no known necessary and sufficient conditions that guarantee convergence. However, sufficient conditions from a study of Keller [3], provide a tool to compare the behavior of some algorithms of the finite difference type. Table 1 presents these conditions for the cases under study and for the most commonly used schemes - explicit (EXP), Crank-Nicholson (C/N) and completely implicit (COMP IMP) methods.

TABLE I
Sufficient conditions for convergence of finite difference methods in solving Eqs. (8) and (16)

EXP	8		$Kh \leq 2q$	$l/h \leq 1/(2q-hK\cos e)$
	16		$Kh \leq 2q$	$l/h \leq 1/2q$
C/N	8	$Kl < 1/2$	$Kh \leq 2q$	$l/h \leq 2/(2q-hK\cos e)$
	16		$Kh \leq 2q$	$l/h \leq 1/q$
COMP IMP	8	$Kl < 1$	$Kh \leq 2q$	
	16		$Kh \leq 2q$	

In the table, h and l stand for space(3) and time increments respectively.

Details can be found in [4]. The third condition, expressing a relation between h and l restricts the use of the first two methods specially if we are to compute the solution with high spatial resolution and a large time horizon. For example using h=.19, K=1. and q=1. the Crank-Nicholson method applied to Eq.(8) require $l < .036$, whereas for the completely implicit method the requirement is $l < 1$.

Under the conditions of table I, the three methods are stable and consistent and the truncation error can be found to be $O(h^2)+O(l)$ for the explicit and completely implicit methods and $O(h^2)+O(l^2)$ for the Crank-Nicholson method.

Because only sufficient conditions for convergence are available, tests were carried out to help in choosing the integration method. It was found that violation of the conditions in the third column of table I could lead to a nonpositive solution. In particular, the Crank-Nicholson method for h=.19, K=1., q=1. and l=.1 gives negative values for the density of Eq.(8), whereas for l=.01<.036 the solution is correct.

The considerations made, led to the decision of using the completely implicit method. The resultant algorithm is presented in the next section.

(3) space is used to denote the variable e troughout this paper

III - NUMERICAL INTEGRATION

The completely implicit approximation relates the function values at four mesh points, three from the line t=j+1 and one from the line t=j. This relation is obtained by replacing the spatial derivatives by finite differences at time t=j+1. Writing down the equations for all spatial grid points, we end up with a matrix equation (see [4] or [5])

$$DP^{n+1} = P^n \qquad (17)$$

where P^j is a vector containing the function values at all the spatial grid points at time t=j and D is a tridiagonal matrix.

Looking at condition (9), it follows that it has to be approximated in order to get a finite support to be used in the computations. However to build an algorithm that can be used for an arbitrary time horizon, the spatial grid must be adaptive, i.e. it must grow with time. This was done by comparing the area under the curve in the last but one 2π interval with the area in the central 2π interval. When the ratio of the two areas is smaller than a certain parameter, the spatial grid is increased by one 2π interval on each side. The function is then initialized with zeros on the new points. The parameter that commands the growth of the grid was found experimentally (see [4]). The initial condition was approximated by a unit area rectangle at the origin. The spatial grid was dimensioned to simplify the computation of the elements of D. Actually, if the points are chosen to be equispaced and identically distributed in each 2π interval, these elements, which are periodic functions (sine and cose), have to be computed in one 2π interval. A further test of the algorithm was done by performing the superposition of Eq.(12) on the numerical solution of Eq.(8)-(9)-(10) and comparing it with the analytic solution of Eq.(13). The square value of the area of the error between them was found to be of the order of 10^{-6}.

IV - DISCUSSION OF THE RESULTS

In this section, the numerical results obtained by integrating Eqs.(8) and (16) are presented. Conclusions on the nature of the phase error are stressed.

The tpdf of the phase error is of the multimodal type. Being a diffusion, the process has a density whose support grows with time. Fig.1 shows the density for two different signal to noise ratios, α=5dB and α=-5dB, pointing out the relation between the diffusion velocity and α. The results presented were obtained for the normalized time instant t/T=2, where T is the mean time between cycle-slips.

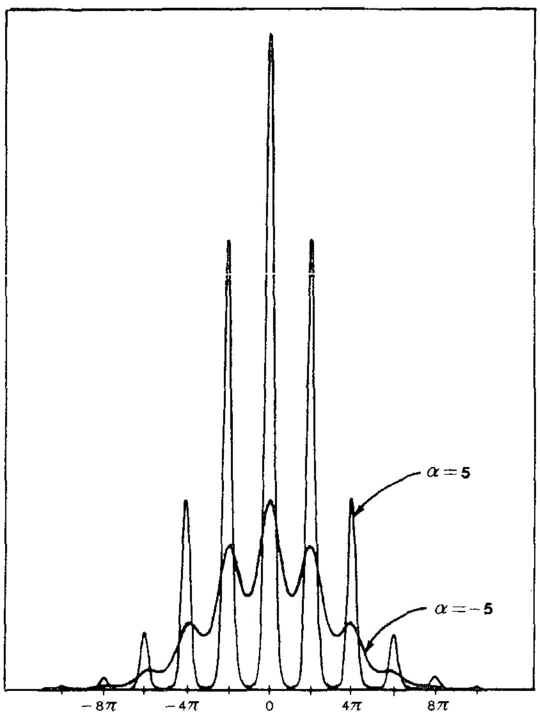

Figure 1

tpdf of the phase error
for α=5,-5dB, and for t/T=2.

When α=5dB the figure suggests that the density can be obtained by repeating the same shape in every 2π interval weighted by appropriate factors.

Consider the decomposition

$$e(t)=\tilde{e}(t)+2\pi N(t) \quad (18)$$

where $\tilde{e}(t)$ is the cyclic phase error defined by Eq.(11) and N(t) is a counting process. If the two processes were independent, then the density of the line error would be a convolution of $\tilde{p}(\tilde{e})$ with a sum of Dirac delta functions standing for the density of N(t), i.e.,

$$p(e,t) = \tilde{p}(\tilde{e}) * \sum N_k(t)\ \delta(e-2k\pi) \quad (19)$$

Looking at Eq.(15), it can be seen that this would be the case if the envelope, E(e,t), were a function constant in each interval of the form $[(2k-1)\pi,(2k+1)\pi]$.

Fig.2 shows the envelope for the cases considered in Fig.1. The staircase function represented corresponds to the actual weights in each 2π interval. It can be seen that when the noise is strong (α=-5dB), the envelope is far from the staircase function leading to the conclusion that the independence assumption doesn´t fit well.

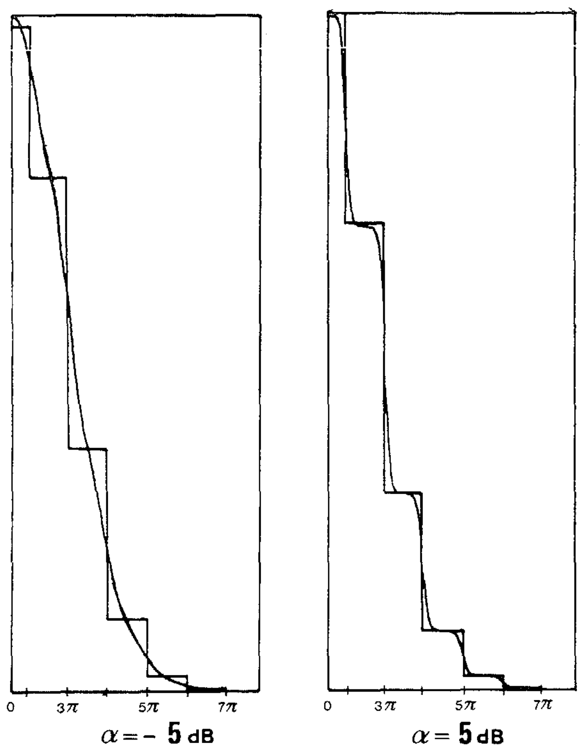

Figure 2

Envelope of Eq.(16)
for α=5,-5dB, and for t/T=2.

In order to compare the shape of the envelope for various signal to noise ratios, Fig.3 presents the envelope in the interval $[2\pi,4\pi]$ normalized by the difference between the level in 2π and the level in 4π.

The curves show that with respect to model of Eq.(18) there exists correlation between $\tilde{e}(t)$ and N(t). However this correlation tends to decrease when α grows.

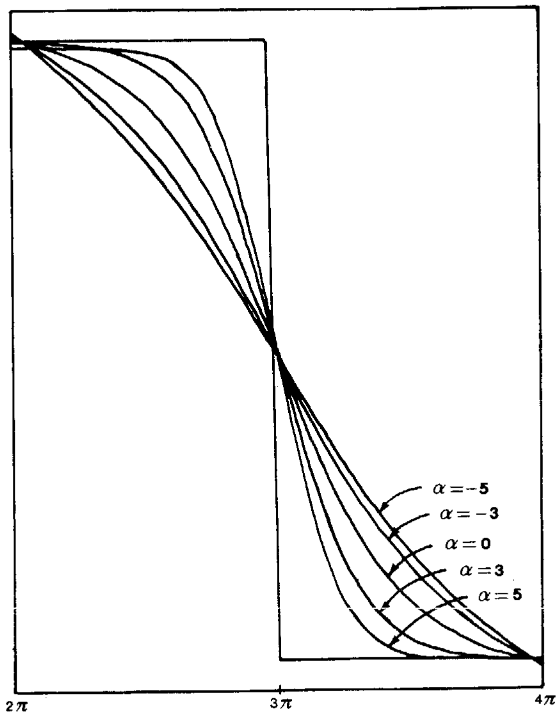

Figure 3

Normalized envelope in $[2\pi, 4\pi]$ for α=-5,-3,0,3,5dB, and for t/T=2.

V - CONCLUSIONS

An algorithm to solve numerically the Fokker-Planck equation associated with the PLL phase error process was developed and a convergence study provided. The difficulties arising from the unbounded support of the error process line density were solved by means of an adaptive space grid.

The density on the line has been factored as the product of an envelope, that satisfies a Kolmogorov Backward partial differential equation, and the steady state solution for the cyclic problem. The solution for the envelope equation shows that there is correlation between the counting process and the cyclic component in the usual decomposition of the PLL error process. This correlation becomes smaller as the signal to noise ratio increases. The present study also shows the difficulties of some of the numerical differencing schemes in integrating the F/P equation. This observation is useful in a more general context, namely that of implementation of optimal nonlinear filters. Loosely interpreting these filters in terms of an iterative sequence of a F/P equation and a multiplicative integral operator, the Crank-Nicholson method should be used with precaution [5] due to its difficulties in preserving the positiveness of the solution for certain values of the statistical parameters.

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