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Some aspects of the two-dimensional nonrecursive Walsh filtering

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RESUME

Cette communication rapporte deux algorithmes du filtrage nonrecursif en deux dimensions (2-D) réalisés à l aide des transformations discrètes de Walsh en deux dimensions. Le premier algorithme profite de la rélation matricielle géneralisé du filtrage, l autre algorithme présenté ici réalise le filtrage en deux dimensions comme la suite des sequences du filtrage scalaire des rangs ou des colonnes de la matrice des données. Dans le premier cas le filtre est determiné par N^4 coefficients (les dimensions de la matrice des donnes sont N×N), dans le deuxiènce par $2N^2$ coefficients qui sont les éléments de deux matrices du filtre.

En plus, le problème de synthèse du filtre bidimensionel dans l espace discret de Walsh est considéré. Dans le premier cas on assume que le filtre bidimensionel est representé par la matrice N×N des coefficients et dans le deuxiènce cas, pour réduire la complexité de calcul, on assume la répresentation du filtre bidimensionel dans le forme de deux matrices diagonales de dimensions N×N. L étude est faite pour le cas des signaux périodiques. Le problème de l optimisation paramètrique du filtre est considéré pour les deux versious du filtre présentés ici dans le but de minimaliser l erreur moyenne quadratique du filtrage.

SUMMARY

The paper is devoted to the two 2-dimensional (2-D) nonrecursive digital filtering algorithms, which are realized by the use of 2-D discrete Walsh transform. The first one results in a generalized matrix relation, according to the second one the 2-D filtering is a sequential application of 1-D filtering to the rows and columns of the N×N data matrix. In the first case the properties of the 2-D filter are determined by its N^4 coefficients. In the second case the properties of the filter are given by the $2N^2$ elements of the two filter matrices. Moreover, the problem of designing of 2-D Walsh domain filter is discussed.

It is assumed, that the 2-D filter is discribed by the N×N filter matrix (in the first case) or, in order to reduce the computational effort, by two diagonal N×N filter matrices (in the second case). The signals, which are to be filtered, are periodic. The problem of optimization of the filter matrices in both above mentioned cases, is considered. The optimization criterion is the minimal value of the mean-square-error of the filtering.

1. Introduction

The discrete Fourier series-transform technique is a classical technique which is used to the filtering of one- and two-dimensional sampled signals [2]. The filtering operation can be also implemented by other transformations than the discrete Fourier transform (DFT) e.g. by the discrete Walsh transform (DWT), which can be computed much faster than the Fourier transform [1].

The DWT does not require any multiplication operations for the Walsh functions take only two values "+1" and "-1". The Walsh functions set as a binary set is directly compatible with digital computers.

In the 2-D nonrecursive filtering process the 2-D component amplitudes are attenuated in the 2-D Walsh domain by the use of some appropriate function, called filter function.

2. The algorithms of 2-D nonrecursive digital filtering in 2-D Walsh domain

A 2-D nonrecursive linear filtering process of a 2-D finite duration sequence $\{v(k,l)\}$ $(k,l=0,1,\ldots,N-1,\ N=2^p$, p-integer) by the use of the 2-D discrete Walsh transform denotes:

1) the transformation of this sequence into the 2-D Walsh domain according to the equation

$$V_W(i,j)=\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} v(k,l)\,\mathrm{wal}(i,k)\,\mathrm{wal}(j,l) \quad (1)$$

$$i,j = 0,1,\ldots,N-1$$

where wal(i,k) and wal(j,l) represent the discrete Walsh functions inthe k and l directions, and i and j are the numbers of zero crossing of the Walsh functions in the k and l directions, respectively.

In matrix notation the formula (1), which determines the 2-D DWT, has the form

$$\underline{V}_W = \frac{1}{N^2}\,\underline{W}\;\underline{V}\;\underline{W}^T \quad (2)$$

where $\underline{V}$ and $\underline{V}_W$ represent N×N data and transform matrices with elements v(k,l) and $V_W(i,j)$,

$\underline{W}$ denotes the N×N Walsh transform matrix with elements wal(j,l) that is "+1" and "-1",

$\underline{W}^T$ - the transposed matrix.

2) The shaping of the 2-D required Walsh spectrum $U_W(m,n)$ by the attenuation of the particular amplitudes $V_W(i,j)$ by the formula

$$U_W(m,n) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} G^W(i,j,m,n)\,V_W(i,j) \quad (3)$$

where $G^W(i,j,m,n)$ is a filter weighting function in 2-D Walsh domain.

3) The inverse transformation of the spectrum $U_W(m,n)$ into the original domain according to the 2-D inverse discrete Walsh transform, which is defined as

$$u(k,l)=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} U_W(i,j)\,\mathrm{wal}(i,k)\,\mathrm{wal}(j,l) \quad (4)$$

or in matrix notation

$$\underline{U} = N^2\,\underline{W}^{-1}\,\underline{U}_W\,(\underline{W}^T)^{-1} \quad (5)$$

where $\underline{W}^{-1}$ is the inverse matrix.

Taking into account, that

$$\underline{W}^{-1} = \frac{1}{N}\underline{W}^T \quad \text{and} \quad \underline{W}^T = \underline{W}$$

equation (5) can be written as

$$\underline{U} = \underline{W}\;\underline{U}_W\;\underline{W} \quad (6)$$

The formulae (1), (3) and (4) determine the general 2-D nonrecursive filtering algorithm of a 2-D sampled signal v(k,l) in 2-D Walsh domain. The block diagram of this algorithm is presented in Fig.1.

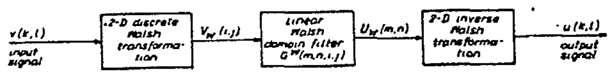

Fig 1 The block diagram of the general two-dimensional (2-D) filtering algorithm in Walsh domain

The number of mathematical operations, which are required to filter N^2 values of the 2-D signal according to the general 2-D filtering algorithm equals $4N^2\log_2 N$

real additions/subtractions and N^4 real multiplications using the fast Walsh transform. It is possible to significantly reduce the computational requirements assuming, that the filter weighting function $G^W(i,j,m,n)$ has the matrix form $[G^W(i,j)]$. Each 2-D spectral component $V_W(i,j)$ of the input data matrix $\underline{V}$ is then weighed individually. Equation (3) will have the following form

$$\hat{U}_W(i,j) = G^W(i,j)V_W(i,j) \qquad (7)$$

where $\hat{U}_W(i,j)$ is the estimate of $U_W(i,j)$. In this case some errors are to be expected. Using the fast Walsh transform only $4N^2\log_2N$ real additions/subtractions and N^2 real multiplications are required for filtering N^2 samples.

Let us note that for the 2-D nonrecursive linear filtering in 2-D Fourier domain the filter weighting function has the matrix form $[G^F(i,j)]$.

The 2-D nonrecursive filtering process can be also treated as a sequential application of 1-D filtering to the rows and columns of N×N data matrix $\underline{V}$.

The block diagram of this kind 2-D filtering algorithm in 2-D Walsh domain is shown in Fig.2.

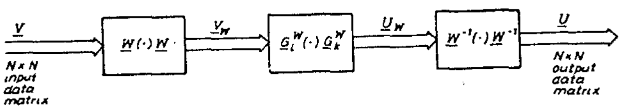

Fig 2 The block diagram of the two-dimensional filtering process in Walsh domain

The corresponding formula can be written as

$$\underline{U} = \underline{W}^{-1}\underline{G}_1^W\underline{W}\ \underline{V}\ \underline{W}\ \underline{G}_k^W\underline{W}^{-1} \qquad (8)$$

where $\underline{G}_k^W$ and $\underline{G}_1^W$ are the Walsh N×N filter matrices for the rows and columns respectively, of the input data matrix $\underline{V}$.

In general case the above mentioned filter matrices contain off-diagonal terms resulting from the fact of equivalence with respect to the Fourier filter matrices, which are diagonal. If the filter matrices $\underline{G}_k^W$ and $\underline{G}_1^W$, which express the properties of the 2-D filter, have a diagonal and constant form, then in the 2-D filtering process some errors are to be expected. However, if the 2-D input signals are known, these errors can be minimized.

In this case the number of mathematical operations, which are required to filter N^2 samples of the 2-D signal equals $4N^2\log_2N$ real additions/subtractions and $2N^2$ real multiplications by the use of the fast Walsh transform.

3. Designing of the 2-D Walsh domain filter

Let the 2-D sampled input signal $v(k,l)$ be a periodic signal, which is a sum of the signal $u(k,l)$ and noise $z(k,l)$ i.e.

$$v(k,l) = u(k,l) + z(k,l)$$
$$k,l=0,\ldots,N-1,\ N=2^p \quad p\text{-integer.}$$

The task of the filter is to extract the discrete signal $u(k,l)$.

In order to reduce the computational effort the filter weighting function $G^W(i,j,m,n)$ is replaced by the filter matrix $[G^W(i,j)]$ or by the two diagonal filter matrices $[G^W(j,j)]$ and $[G^W(i,i)]$. The 2-D filtering process is realized according to equations (7) or (8).

The optimal filter matrices $[G^W(i,j)]$ or $[G^W(j,j)]$ and $[G^W(i,i)]$ will be now calculated. The condition of optimality is the minimization of the mean-square-error of the filtering

$$\overline{\varepsilon^2} = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\left[\hat{u}(k,l) - u(k,l)\right]^2 \qquad (9)$$

where $\hat{u}(k,l)$ is the estimate of the signal $u(k,l)$.

According to the Parseval's theorem the mean-square-error (9) can be written in the following form

$$\overline{\varepsilon^2} = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\left[\hat{U}_W(i,j) - U_W(i,j)\right]^2 \qquad (10)$$

where

$$\hat{U}_W(i,j) = G^W(i,j)V_W(i,j) \qquad (11)$$

or

$$\hat{U}_W(i,j) = G^W(j,j)V_W(i,j)G^W(i,i) \quad (12)$$

The optimal values of $G^W(i,j)$ or $G^W(j,j)$ and $G^W(i,i)$ result from the following conditions

$$\frac{\partial \overline{\varepsilon^2}}{\partial G^W(i,j)} = 0 \quad (13)$$

$$\frac{\partial \overline{\varepsilon^2}}{\partial G^W(j,j)} = 0$$

$$\frac{\partial \overline{\varepsilon^2}}{\partial G^W(i,i)} = 0 \quad (14)$$

Taking into account the expressions (10), (11) and (12) we find that the equation (13) is

$$\left[G^W(i,j)V_W(i,j) - U_W(i,j)\right]V_W(i,j)=0 \quad (15)$$

and equations (14) have the following form

$$\sum_{i=0}^{N-1}\left[G^W(j,j)V_W(i,j)G^W(i,i) - U_W(i,j)\right]V_W(i,j)G^W(i,i)=0$$

$$\sum_{j=0}^{N-1}\left[G^W(j,j)V_W(i,j)G^W(i,i) - U_W(i,j)\right]V_W(i,j)G^W(j,j)=0 \quad (16)$$

From equation (15) we have

$$G^W(i,j) = \frac{U_W(i,j)}{V_W(i,j)} \quad (17)$$

Equations (16) can be written as the following system of 2N equations with 2N unknown values $x_0,\ldots,x_{N-1}$ and $y_0,\ldots,y_{N-1}$

$$x_0(a_{00}y_0^2+\ldots+a_{N-1,0}y_{N-1}^2)=b_{00}y_0+\ldots+b_{N-1,0}\cdot y_{N-1}$$

....

$$x_{N-1}(a_{0,N-1}y_0^2+\ldots+a_{N-1,N-1}y_{N-1}^2) = b_{0,N-1}y_0+\ldots+b_{N-1,N-1}y_{N-1}$$

$$y_0(a_{00}x_0^2+\ldots+a_{0,N-1}x_{N-1}^2) = b_{00}x_0+\ldots+b_{0,N-1}x_{N-1}$$

....

$$y_{N-1}(a_{N-1,0}x_0^2+\ldots+a_{N-1,N-1}x_{N-1}^2) = b_{N-1,0}x_0+\ldots+b_{N-1,N-1}x_{N-1}$$

where $x_j=G^W(j,j)$ $y_i=G^W(i,i)$

$a_{ij}=V_W^2(i,j)$ $b_{ij}=U_W(i,j)V_W(i,j)$.

This system of equations can be solved by the use of the iteration method.

4. Conclusions

2-D Walsh transform enable us to realize a simplified 2-D nonrecursive digital filter. The number of required mathematical operations can be reduced but some error signals are to be expected. It should be underlined that in general the entries of filter matrices $[G^W(i,j)]$, $[G^W(j,j)]$ and $[G^W(i,i)]$ take different values for various 2-D signals even if the same 2-D frequency response is assumed. It can be seen, that the filter matrices depend on the 2-D input spectra.

It must be noted also, that in some particular cases the minimum value of the error can be equal zero, namly if the Walsh spectra $U_W(i,j)$ and $Z_W(i,j)$ of the signal $u(k,l)$ and noise $z(k,l)$ do not overlap.

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