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LOWER BOUNDS ON TARGET LOCALIZATION ERRORS FOR VARIOUS SIGNAL AND NOISE CHARACTERISTICS

AZIZUL H. QUAZI

NAVAL UNDERWATER SYSTEMS CENTER, NEW LONDON, CT 06320 USA

RESUME

SUMMARY

In a passive sonar system, a target may be localized using the differential time delays of the target's signal arrivals at spatially separated sensors. The accuracy of this time delay estimation (TDE) procedure depends on the signal-to-noise ratio (SNR), observation time, effective bandwidth, and center frequency of the signal and noise and their spectral characteristics. Here we investigate the effects of the signal and noise characteristics on the lower bounds on TDE and on target localization, i.e., bearing and range estimation capability. We also investigate the effects on the nonlinear transformation from TDE to bearing and range estimation. The analysis results indicate that the lower bounds on time delay errors depend heavily on the signal and noise characteristics. However, the lower bounds remain constant if the signal and noise spectral slopes are the same and all other parameters such as SNR, observation time, and bandwidth are held constant. The bias due to the nonlinear transformation introduced in bearing estimation is negligible, but the bias in range estimation is always positive and is proportional to the cube of the true range and inversely proportional to the square of the effective base leg length. The root mean square error of the range estimate is evaluated in terms of the true range, the base leg length, the variances of the bearing errors, and the minimum variances of the range errors.

LOWER BOUNDS ON TARGET LOCALIZATION ERRORS FOR VARIOUS SIGNAL AND NOISE CHARACTERISTICS

INTRODUCTION

In a passive sonar system, a target may be localized by using the differential time delays of target signal arrivals at spatially separated sensors. The localization of a target involves estimating its bearing and range. As shown in Figure 1, the difference in arrival times of the target's signal at A and B and at B and C determines the bearing angles, θ_1 and θ_2. Once θ_1 and θ_2 and the array separations, L_1 and L_2, are known, the range of the target can be determined (see Figure 1).

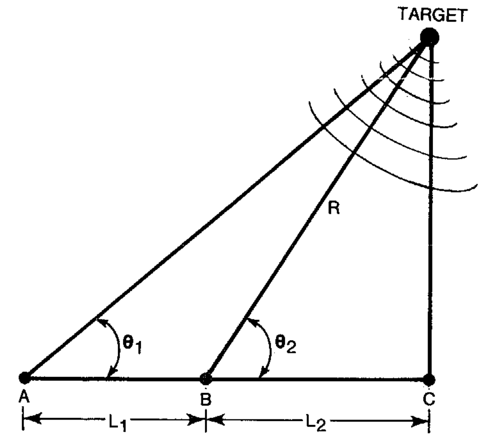

Fig. 1. Target's bearing and range estimation

The time difference utilized in bearing estimation is obtained by cross-correlating the received waveforms at two spatially separated points (Figure 1: A,B and B,C) and measuring the time displacement of correlogram peaks (Figure 2). The received signals are corrupted with noise, causing an inaccuracy in the measurement of correlation peaks that affects the bearing angles, θ_1 and θ_2, and range estimates. The unknown time difference or differential time delay is estimated as shown in Figure 2.

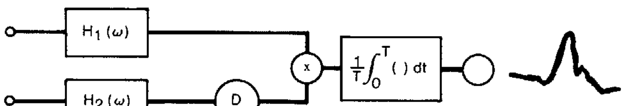

Fig. 2. Time delay estimation using generalized cross correlator.

The accuracy of the differential time delay estimation (TDE) depends on the signal-to-noise ratio (SNR), observation time, effective bandwidth, and center frequency of the signal and noise and their spectral characteristics. In this paper, we investigate the effects of signal and noise spectral characteristics on the lower bounds on TDE and, hence, on bearing and range estimation capabilities. Furthermore, we also investigate the effects of the nonlinear transformation from TDE to bearing and range estimation.

TIME DELAY ERRORS

Hahn[1] has shown that the variance σ_D^2 of a TDE procedure that is assumed to be unbiased is

$$\sigma_D^2 = \frac{2\pi}{T} \cdot \frac{\int_{-\infty}^{\infty} \omega^2 \left|H(\omega)\right|^4 \left[N_1(\omega)N_2(\omega) + S(\omega)\left\{N_1(\omega) + N_2(\omega)\right\}\right] d\omega}{\left[\int_{-\infty}^{\infty} \omega^2 \left|H(\omega)\right|^2 S(\omega) d\omega\right]^2} \qquad (1)$$

where

$S(\omega)$, $N_1(\omega)$, and $N_2(\omega)$ = the power density spectra of signal $s(t)$, noise $n_1(t)$, and noise $n_2(t)$, respectively;

T = observation time in seconds;

f = frequency in Hertz; and $\omega = 2\pi f$;

$H(\omega)$ = the voltage transfer function of the prefilter (Figure 2).

The variance of the TDE is minimum when the filter power transfer function satisfies

$$\left|H(\omega)\right|^2 = \frac{S(\omega)}{N_1(\omega)\, N_2(\omega)} \left[1 + \frac{S(\omega)}{N_1(\omega)} + \frac{S(\omega)}{N_2(\omega)}\right]^{-1} . \qquad (2)$$

Substituting the value $\left|H(\omega)\right|^2$ into equation (1) yields[2,3]

$$\sigma_D^2 = \frac{\pi}{T} \left\{ \int_0^{\infty} \omega^2 \frac{\dfrac{S^2(\omega)}{N_1(\omega)\, N_2(\omega)}}{1 + \dfrac{S(\omega)}{N_1(\omega)} + \dfrac{S(\omega)}{N_2(\omega)}} d\omega \right\}^{-1} . \qquad (3)$$

Now, if the signal and noise spectra are constant over the band extending from f_1 to f_2 Hz or the spectral fall-off rates are equal, then equation (3) may be written as[4]

$$\sigma_D^2 = \frac{1}{8\pi^2} \frac{1}{TW} \frac{1}{f_{rms}^2} \frac{1 + SNR_1 + SNR_2}{SNR_1 \cdot SNR_2} , \qquad (4)$$

where

SNR_1 and SNR_2 = power SNR's over the band W Hz,

$$f_{rms} = f_o \sqrt{1 + \frac{W^2}{12 f_o^2}} \text{ Hz}, \quad W = f_2 - f_1 \text{ Hz}$$

$$f_o = f_1 + \frac{W}{2} = f_2 - \frac{W}{2} \text{ Hz.}$$

Therefore, the standard deviation of the time errors is given by

$$\sigma_D = \frac{1}{2\pi} \frac{1}{\sqrt{2TW}} \frac{1}{f_{rms}} \frac{\sqrt{1 + 2\, SNR}}{SNR} , \quad SNR_1 = SNR_2 = SNR \qquad (5a)$$

$$\approx \frac{1}{2\pi} \frac{1}{\sqrt{2TW}} \frac{1}{f_{rms}} \frac{1}{SNR} , \quad \text{for } SNR \ll 1 \qquad (5b)$$

$$\approx \frac{1}{2\pi} \frac{1}{\sqrt{2TW}} \frac{1}{f_{rms}} \frac{1}{\sqrt{SNR/2}}\,, \text{ for SNR} >> 1. \qquad (5c)$$

The consequence of different and unequal signal and noise spectral fall-off rates on σ_D are described elsewhere.[4,5]

Notice that equation (5) shows that σ_D is independent of the true differential time delay and depends only on the SNR, the observation time, and the signal and noise characteristics.

TARGET LOCALIZATION ERRORS

Target localization involves bearing and range determination. Once the differential time delays are known, the bearing angles, θ_1 and θ_2, are calculated as follows:

$$\cos\theta_1 \approx \frac{cD_1}{L_1}\,, \text{ and } \cos\theta_2 \approx \frac{cD_2}{L_2}\,, \qquad (6)$$

where

D_1 and D_2 = the differential time delays of target signal arrivals at sensors A and B and B and C, respectively;

c = the sound speed; and

L_1 and L_2 = the lengths of the baseline (Figure 1).

If θ_1, θ_2, and L_1 are known accurately, then the ranges, R, of the target can also be determined accurately.

Now let $D_1 = D_{01} + \delta D_1$, where D_{01} is the true differential time delay and δD_1 ($<< D_{01}$) is the random error in the time delay. We assume that the mean time delay error is zero; i.e.,

$$\overline{\delta_{D1}} = 0 \quad \text{and} \quad \overline{D_1} = D_{01}\,.$$

We shall calculate the bias in the estimation of θ_1 (similar arguments hold for θ_2) as a function of L_1, θ_{01}, and σ_{D1}, where θ_{01} and σ_{D1} are the mean bearing angle and standard deviation of time delay errors, respectively.

Expanding the differential time delay, D_1, about D_{01} using a Taylor series, we see[6] that the expected value of the bearing angle estimate θ_1 is

$$E\left[\hat{\theta}_1\right] \approx \theta_{01} + \frac{d^2\theta_1}{dD_1^2}\frac{\sigma_{D1}^2}{2} + \ldots \qquad (7)$$

or

$$E\left[\hat{\theta}_1\right] - \theta_{01} \approx \frac{d^2\theta_1}{dD_1^2}\frac{\sigma_{D1}^2}{2}$$

and

$$\cos\theta_1 = \left(\frac{c}{L_1}\right)D_1\,; \qquad (8)$$

so,

$$\frac{d^2\theta_1}{dD_1^2} = -\left(\frac{c}{L_1}\right)^2 \frac{\cos\theta_1}{\sin^3\theta_1}\,.$$

Therefore, the bias in the bearing angle at θ_{01} is

$$\theta_{B1} = E\left[\hat{\theta}_1\right] - \theta_{01}$$

$$= -\left(\frac{c}{L_1}\right)^2 \frac{\cos\theta_{01}}{\sin^3\theta_{01}} \frac{\sigma_{D1}^2}{2}$$

$$= -\left(\frac{c}{L_1 \sin\theta_{01}}\right)^2 \frac{\cos\theta_{01}}{\sin\theta_{01}} \frac{\sigma_{D1}^2}{2}\,. \qquad (9)$$

Similarly, the variance of the bearing angle errors using a Taylor expansion can be written as[6]

$$\sigma\theta_{01}^2 = \left(\frac{d\theta_1}{dD_1}\right)^2 \sigma_{D1}^2$$

$$= \left(\frac{c}{-L_1 \sin\theta_{01}}\right)^2 \sigma_{D1}^2$$

$$\sigma\theta_{01}^2 = \frac{c^2}{L_1^2 \sin^2\theta_{01}} \sigma_{D1}^2\,. \qquad (10)$$

Therefore, the standard deviation of the bearing angle error is

$$\sigma\theta_{01} = \frac{c}{L_1 \sin\theta_{01}} \sigma_{D1}\,. \qquad (11)$$

The bias θ_{B1} may be expressed in terms of the variance of bearing errors. Equation (9) may be simplified as follows:

$$\theta_{B1} = -\left(\frac{c\,\sigma_{D1}}{L_1 \sin\theta_{01}}\right)^2 \frac{1}{2}\frac{\cos\theta_{01}}{\sin\theta_{01}}$$

$$= -\left(\sigma\theta_{01}\right)^2 \frac{\cot\theta_{01}}{2}\,. \qquad (12)$$

Notice that the bias (equation (12)) error is proportional to the variance of bearing error and $\cot\theta_{01}$, and is zero at broadside ($\theta_{01} = 90°$). The bias could be positive or negative, depending on the bearing of the target relative to broadside.

The bearing angles, θ_1 and θ_2, are determined from the differential time delays. Once θ_1 and θ_2 and base leg length, L_1, are known, the range of the target, R, may be determined as follows:

$$R = L_1 \frac{\sin\theta_1}{\sin(\theta_2 - \theta_1)}$$

$$\approx L_1 \frac{\sin\theta_1}{\theta_2 - \theta_1} \qquad (13)$$

when $|\theta_2 - \theta_1|$ is very small.

Now, if θ_1, θ_2, and L_1 are exactly measured or known, then R is exactly known. But there will be errors in θ_1 and θ_2 that cause an error in the range estimate. R is nonlinearly related to θ_1 and θ_2 for a fixed L_1, and this nonlinear relationship will cause a bias in the range estimate.

The expected value of the range and the variance of the range may be written as[6,7]

$$E[\hat{R}(\hat{\theta}_1, \hat{\theta}_2)] \simeq R + \frac{1}{2}\left\{\frac{\partial^2 R}{\partial\theta_1^2}\right\}\sigma_{\theta 1}^2 + \frac{1}{2}\left\{\frac{\partial^2 R}{\partial\theta_2^2}\right\}\sigma_{\theta 2}^2 \qquad (14)$$

and

$$\sigma_R^2 \simeq \left(\frac{\partial R}{\partial\theta_1}\right)^2 \sigma_{\theta 1}^2 + \left(\frac{\partial R}{\partial\theta_2}\right)^2 \sigma_{\theta 2}^2 \quad . \qquad (15)$$

Assuming that the average of the cross product due to θ_1 and θ_2 is zero and R is the true range, $\sigma_{\theta 1}^2$ and $\sigma_{\theta 2}^2$ are the variances of angles θ_1 and θ_2, respectively. If $L_1/R \ll 1$, it can be shown that the bias in the range estimate is

$$R_B \simeq E[\hat{R}(\hat{\theta}_1, \hat{\theta}_2)] - R$$

$$= \frac{R^3}{L_1^2 \sin^2\theta_{01}}\left\{1 + \frac{L_1\cos\theta_{01}}{R} - \frac{L_1^2}{R^2}\sin^2\theta_{01}\right\}\sigma_{\theta 1}^2$$

$$+ \frac{R^3}{L_1^2 \sin^2\theta_{01}}\sigma_{\theta 2}^2$$

$$\simeq \frac{R^3}{L_1^2 \sin^2\theta_{01}}\left(\sigma_{\theta 1}^2 + \sigma_{\theta 2}^2\right) \quad . \qquad (16)$$

Therefore, the normalized bias is

$$R_B/R = \frac{R^2}{L_1^2 \sin^2\theta_{01}}\left(\sigma_{\theta 1}^2 + \sigma_{\theta 2}^2\right) \quad . \qquad (17)$$

Equation (17) shows that the normalized bias R_B/R (or fractional bias range error) is proportional to the square of the ratio of the true range and effective base leg length. The bias error will be smaller if the ratio of true range and effective base leg length is smaller. We have assumed in equation (16) that $\cos\theta_1 \ll 1$, which implies that the target is located close to broadside of the linear array (Figure 1).

Similarly, the variance of the range error is

$$\sigma_R^2 \simeq \left\{\frac{R^2}{L_1\sin\theta_{01}} + \frac{R^2}{L_1}\frac{\cos\theta_{01}}{\sin\theta_{01}}\right\}^2 \sigma_{\theta 1}^2$$

$$+ \left\{-\frac{R^2}{L_1\sin\theta_{01}}\right\}^2 \sigma_{\theta 2}^2$$

$$\simeq \frac{R^4}{L_1^2\sin^2\theta_{01}}\left(\sigma_{\theta 1}^2 + \sigma_{\theta 2}^2\right) \quad . \qquad (18)$$

Therefore, the standard deviation of the range error is

$$\sigma_R \simeq \frac{R^2}{L_1\sin\theta_{01}}\sqrt{\sigma_{\theta 1}^2 + \sigma_{\theta 2}^2} \quad . \qquad (19)$$

Now, if $L_1 = L_2 = L$, $\theta = \theta_1 = \theta_2$, and $\sigma_{D1} = \sigma_{D2} = \sigma_D$,

then $$\sigma_R = \frac{R^2}{L\sin\theta}\sqrt{2\sigma_\theta^2}$$

$$= \frac{R^2}{L^2\sin^2\theta}\,C\sqrt{2}\sigma_D$$

$$= \sqrt{2}\,C\,\frac{R^2}{L^2\sin^2\theta}\,\sigma_D . \qquad (20)$$

Theriault and Zeskind[7] have found similar results.

Now from equation (19)

$$\sigma R/R = \frac{R}{L\sin\theta}\sqrt{\sigma_{\theta 1}^2 + \sigma_{\theta 2}^2} \quad ,$$

but

$$R_B/R = \frac{R^2}{L^2\sin^2\theta}\left(\sigma_{\theta 1}^2 + \sigma_{\theta 2}^2\right) \qquad (21)$$

$$R_B/R = \left(\frac{\sigma R}{R}\right)^2$$

or

$$\frac{\sigma_R}{R} = \sqrt{\frac{R_B}{R}} \quad . \qquad (22)$$

Equation (22) indicates that the fractional bias range error can be determined from the fractional standard deviation of range errors. Theriault and Zeskind[7] have also found similar results.

Bias in the range estimate introduces an additional error. This bias is a function of true range. Therefore, the mean square error of the range estimate may not be predicted as with an unbiased estimate via the Cramer-Rao bound. Kendall, Stuart, and others[8,9] have shown, in the case of biased estimates, that the mean square error of the range, R, is bounded according to

$$MSE(\hat{R}) \geq \left(1 + \frac{dR_B}{dR}\right)^2 \sigma_R^2 \quad , \qquad (23)$$

where σ_R^2 is the unbiased range variance. Therefore,

$$MSE(\hat{R}) \geq \left(1 + 3\frac{\sigma_R^2}{R^2}\right)^2 \sigma_R^2 \quad . \qquad (24)$$

Equation (24) shows that, in the case of biased estimates, the bound on the mean square range error can be calculated from the variance of the unbiased estimate.

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DISCUSSION OF ANALYTICAL RESULTS

We shall now discuss the analytical results obtained in the previous section by means of an example. For the example, we have selected the signal and noise spectra to have a bandwidth of 500 Hz and a center frequency of 1500 Hz. Further, we have assumed that the signal and noise spectral fall-off rates over the band are the same. The range of SNR's is -25 to 0 dB. Figure 3 shows the standard deviation of the time delay error as a function of the average SNR, with observation time T = 20 and 60 seconds as a parameter (equation 5a). It is seen that σ_D decreases with increasing SNR, as expected. With increasing observation time, σ_D decreases. Notice that σ_D is independent of true target position. The standard deviation of the bearing error as a function of average SNR ($SNR_1 = SNR_2 = SNR_{av}$) is presented in Figure 4. Here we have assumed that the length of the baseline is 500 feet. The plots are valid only for a broadside target. Away from broadside, σ_D will increase by a factor $1/\sin\theta$, where θ = bearing angle ($\theta = 90°$ corresponds to broadside).

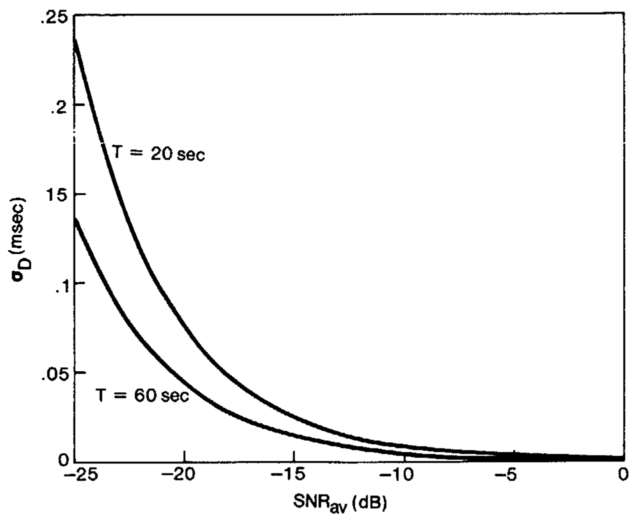

Fig. 3. Standard deviation of time delay errors as a function of average SNR. W = 500 Hz, f_o = 1500 Hz.

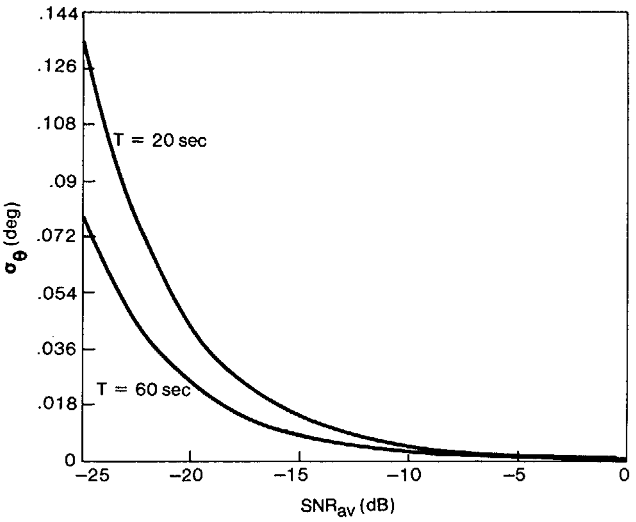

Fig. 4. Standard deviation of bearing error as a function of average SNR. $L_1 = L_2 = L$ = 500 ft.

Figure 5 shows the bias in the range estimate as a function of true range, at broadside, 15°, and 30° off broadside. Observe that the bias is proportional to the cube of true range and is inversely proportional to the effective base leg length squared $(L \sin\theta)^2$ for a constant SNR and observation time. The range bias is always positive. In other words, the true range is often overestimated.

Figure 6 shows the standard deviation of range errors as a function of true range at various bearing angles. σ_R is proportional to the square of the range and inversely proportional to the effective base leg length. Therefore, increasing the baseline would decrease the range error. For example, the range error at a true range of 20 kyd is about σ_R = 2500 yards at broadside. The root-mean-square error as a function of range at different bearing angles is shown in Figure 7.

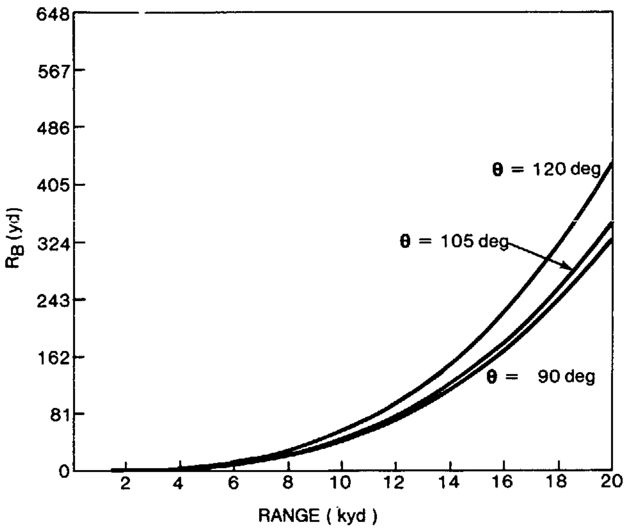

Fig. 5. Bias in range as a function of time range at SNR_{av} = -20 dB, T = 20 sec, L = 500 ft.

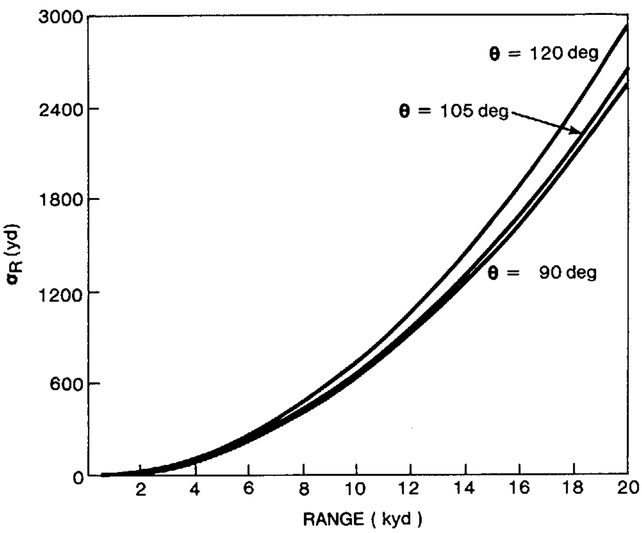

Fig. 6. Standard deviation of range error as a function of true range at SNR_{av} = -20 dB, T = 20 sec, L = 500 ft.

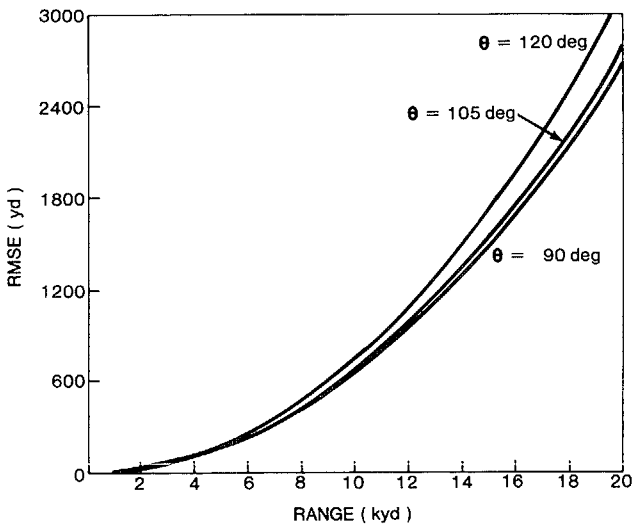

Fig. 7. Root mean square error as a function of true range at SNR_{av} = -20 dB, T = 20 sec, L = 500 ft.

CONCLUSIONS

We have derived simple expressions that yield the standard deviation of time delay errors as a function of SNR, observation time, and signal and noise spectral characteristics. Also, we have attempted to derive simple equations that give the bias and the standard deviation of bearing and range estimates of the target. The bias errors in bearing and range estimation are due to a nonlinear transformation. The analysis results show that the bias in the bearing estimate is proportional to the variance of bearing errors and cot θ, where θ is the true bearing angle; i.e., if the variance is known, the bias can be calculated. The bias in the range estimate is always positive and proportional to the cube of the true range and inversely proportional to the square of the effective base leg length. Therefore, the fractional range bias, i.e., the bias normalized with respect to the true range, can be calculated from the range estimate variance (unbiased range estimate). The biases in bearing and range estimates are inherent and due to the nonlinear transformation and cannot be removed easily.

We have already discussed the effects of time delay errors on bearing and range estimation of a target. We summarize our observations next:

(1) The standard deviation of time delay errors depends only on the SNR, observation time, and signal and noise characteristics. It is independent of true differential time delay; i.e., it is independent of the actual target location.

(2) The bearing errors that include bias and variance depend on the time delay errors, length of the baseline, sound speed, and true bearing angle. The bias in bearing due to the nonlinear transformation depends on the unbiased variance estimate and true bearing angle. In other words, the bias could be determined if the unbiased variance and true bearing angle are known. The bias is zero at broadside. Contingent upon the true bearing angle, the bias could be positive or negative; i.e., the bearing could be overestimated or underestimated, depending upon the true bearing. The unbiased variance estimate depends on the true bearing, length of the baseline, sound speed, and the variance of the time delay errors.

(3) The range errors that include the bias and variance depend on the true range, length of the baseline, and the variance of bearing errors. The range bias depends on the unbiased variance estimate and the true range. Therefore, the range bias can be simply calculated from the unbiased variance estimate and true range. The bias in the range estimate is always positive, which implies that the range is often overestimated. The bound on the mean square error can be predicted from the unbiased variance estimate and the true range.

(4) Owing to the inherent nature of this target localization technique utilizing wavefront curvature, the biases in the bearing and range estimates cannot be totally eliminated. However, they can be reduced by increasing the observation time.

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