ON INSTANTANEOUS FREQUENCY AND CYCLOSTATIONARITY

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RESUME

On présente dans cet article les expressions dérivées pour les moments de la fréquence instantanée d'un signal aléatoire cyclostationnaire. Les résultats montrent que le signal cyclostationnaire n'est pas seulement caracterisé par la périodicité de ses moments statistiques, mais aussi par celle des moments de sa fréquence instantanée. Cette caractérisation paraît valide dans le cas d'une situation unique, aussi bien que dans le cas d'une situation de multi-periodicité, où le premier moment de la fréquence instantanée peut être représenté dans la forme α cyclique décomposée. Pour prouver cette qualitée générale on derive les expressions pour les moments de plus haut ordre de la fréquence instantanée et on commente les conditions nécessaires pour leur existance.

SUMMARY

In this paper we derive expressions for the moments of instantaneous frequency of cyclostationary random signal. Our results show that cyclostationary signal is not only characterized by the periodicity of its statistical moments but, can be also characterized by the periodicity of the moments of its instantaneous frequency. This characterization appears to be valid for either single or multi-periodicity situation, in which case the first moment of instantaneous frequency can be represented in an α-cyclic decomposed form. To prove this general property we also derive expressions for the higher-order moments of instantaneous frequency and discuss the necessary condition for their existence.

I. INTRODUCTION

In this paper we present a statistical study about the instantaneous frequency of a cyclostationary random signal. Instantaneous frequency is among the basic topics in signal theory and, the concept of it has been a subject of considerable discussion in past as well as some recent literature [1-9]. Although cited references as well as the rest of literature not addressed here, give rather exhaustive treatment of the subject, there were few papers to considered nonstationary processes. The main reason was probably the non-existence of appropriate theory as well as the luck of wider interest for that subject. It was only after the study published on Wigner-Ville distribution [10,11] for finite energy signals and its generalization for nonstationary signals [12], that a new and useful theory was available. In [12] a conjoint time-frequency representation of harmonizable nonstationary random signal has been defined and, exploited for the study of the properties of instantaneous frequency and random group delay. Expressions has been given for their expectation and variance without assuming the narrow band condition or stationarity of the random signal.

In this paper we consider a special class of nonstationary processes namely cyclostationary random signal and, study the properties of corresponding instantaneous frequency. We base our consideration mostly on the theory of cyclostationary signals developed by W.Gardner [13].

II. INSTANTANEOUS FREQUENCY

Let $s(t)$ be a real time-dependent signal and $x(t)$ the analytic signal associated with $s(t)$. Then by definition $x(t)=s(t)+j\check{s}(t)$, where $\check{s}(t)$ denotes the Hilbert transform of

$s(t)$, i.e.

$$\check{s}(t) = V.P. \int_{-\infty}^{+\infty} \frac{s(\tau)}{\pi(t-\tau)} dt \qquad (1)$$

where V.P. states for the Cauchy's principal value. The main reason for introducing the analytic signal is that instantaneous frequency in general is uniquely defined for complex signals only and it is proved in [14] that the theory of analytic signal can be also extended to the case of random signal. If $x(t)$ is differentiable in the mean square sense [15], its instantaneous frequency denoted by $f_i(t)$ can be expressed as

$$f_i(t) \triangleq \frac{Im[x'(t)x^*(t)]}{2\pi|x(t)|^2} \qquad (2)$$

where prime states for the first derivative and asterisk for the complex conjugation. Our initial assumption about $x(t)$ implies that (2) is a random function of time too, that can be characterized by its statistical moments. Using the same definition as in [12] the mean of instantaneous frequency denoted by $E\{f_i(t)\}$ can be expressed as

$$E\{f_i(t)\} = \frac{E\left\{Im[\frac{\partial}{\partial t}R_{xx}(t,s)|_{s=t}]\right\}}{2\pi\ var[x(t)]} \qquad (3)$$

where $R_{xx}(t,s) = E\{x(t)x^*(s)\}$. We note that (3) is defined without assuming narrow band condition or stationarity of $x(t)$. If we further assume that $x(t)$ is a cyclostationary random signal alternative expression for (3) can be obtained that reveal an interesting property. We shall say that a phenomenon under study, a cyclostationary random signal $x(t)$, features second order periodicity with cycle frequency α, if

$$R_x^{\alpha}(\tau) = \lim_{z\to+\infty} \frac{1}{z} \int_{-z/2}^{+z/2} R_x(t,\tau)\ e^{-j2\pi\alpha t} dt \qquad (4)$$

$$R_x(t,\tau) = E\{x(t)x^*(t-\tau)\}$$

exists as a function of time-lag τ and is not identically equal to zero; (4) shall be referred to as a cyclic autocorrelation function [13] and can be also interpreted as a coefficient function in generalized Fourier series representation

$$R_x(t,\tau) = \sum_{\alpha} R_x(\tau)\ e^{j2\pi\alpha t} \qquad (5)$$

The Fourier transform

$$S_x^{\alpha}(f) = \int_{-\infty}^{+\infty} R_x^{\alpha}(\tau) e^{-j2\pi f\tau} d\tau \qquad (6)$$

is called a cyclic spectral density function [13] or cyclic spectrum. It easily follows from (6) that (5) can be also expressed as

$$R_x(t,\tau) =$$

$$= \sum_{\alpha} R_x^{\alpha}(\tau)\ e^{j2\pi\alpha t} = \sum_{\alpha} \int_{-\infty}^{+\infty} S_x^{\alpha}(f) e^{j2\pi f\tau} e^{j2\pi\alpha t} df$$

$$= \int_{-\infty}^{+\infty} S_x(t,f) e^{j2\pi f\tau} df \qquad (7)$$

where $S_x(t,f)$ is a spectral periodic function of $x(t)$. From (3) and by use of (7) it is easily shown (see Appendix) that a numerator factor in (3) can be evaluated as

$$E\{Im(x'(t)x^*(t))\} = \frac{1}{j} \sum_{\alpha} \frac{d}{d\tau} R_x^{\alpha}(\tau)\big|_{\tau=0} e^{j2\pi\alpha t}$$

$$= \sum_{\alpha} \int_{-\infty}^{+\infty} f S_x^{\alpha}(f) e^{j2\pi\alpha t} df = \int_{-\infty}^{+\infty} f S_x(t,f) df \qquad (8)$$

Assuming $E\{x(t)\}=0$, the variance of $x(t)$ is given by

$$var[x(t)] = \sum_{\alpha} R_x^{\alpha}(o) e^{j2\pi\alpha t}$$

$$= \sum_{\alpha} \int_{-\infty}^{+\infty} S_x^{\alpha}(f) e^{j2\pi\alpha t} df = \int_{-\infty}^{+\infty} S_x(t,f) df \qquad (9)$$

Substituting the corresponding factors from (8) and (9) into (3) yields the following expression for the mean of instantaneous frequency

$$E\{f_i(t)\} = \frac{1}{j2\pi} \frac{\sum_{\alpha} \frac{d}{d\tau} R_x^{\alpha}(\tau)\big|_{\tau=0} e^{j2\pi\alpha t}}{\sum_{\alpha} R_x^{\alpha}(o) e^{j2\pi\alpha t}} \qquad (10a)$$

$$= \frac{\sum_{\alpha} \int_{-\infty}^{+\infty} f S_x^{\alpha}(f) e^{j2\pi\alpha t} df}{\sum_{\alpha} \int_{-\infty}^{+\infty} S_x^{\alpha}(f) e^{j2\pi\alpha t} df} \qquad (10b)$$

$$= \frac{\int_{-\infty}^{+\infty} f S_x(t,f) df}{\int_{-\infty}^{+\infty} S_x(t,f) df} \qquad (10c)$$

Here we have assumed that $R^{\alpha}(\tau)$ is a continuous function of time-lag $\tau \in \mathbb{R}$ possessing a first derivative for $\tau=0$, thus excluding anomalous condition. In the case of a pure cyclostationary signal $x(t)$ that is characterized by a single period T expressions (10a) can be modified as follows

$$E\{f_i(t)\} = \frac{1}{j2\pi} \frac{\sum_{m} \frac{d}{d\tau} R_x^{m/T}(\tau)\big|_{\tau=0} e^{j2\pi mt/T}}{\sum_{m} R_x^{m/T}(o) e^{j2\pi mt/T}} \qquad (11)$$

which reveals that (11) is a periodic function of time t with the same period T as $R_x(t,\tau)$ in (7), i.e.

$$E\{f_i(t+T)\} = E\{f_i(t)\} \qquad (12)$$

Since the numerator in (11), can be thought as a convenient weighting factor it implies series representation

$$E\{f_i(t)\} = \sum_{m} f_i^{m/T}\ e^{j2\pi mt/T} \qquad (13)$$

given

$$\frac{d}{d\tau} R_x^{m/T}(\tau)\big|_{\tau=0} \neq 0, \quad m \neq 0$$

since otherwise (13) reduces to a single real constant f^{o}. Reasoning the same way for almost periodic cyclostationary signal $x(t)$ we obtain the more general expression

$$E\{f_i(t)\} = \sum_{\alpha} f_i^{\alpha} e^{j2\pi\alpha t} \tag{14a}$$

$$f_i^{\alpha} \triangleq \lim_{Z\to+\infty} \frac{1}{Z}\int_{-Z/2}^{+Z/2} E\{f_i(t)\}\, e^{-j2\pi\alpha t}dt$$

$$\frac{d}{d\tau}R_x^{\alpha}(\tau)\Big|_{\tau=0} \neq 0, \quad \alpha \neq 0 \tag{14b}$$

where (14a) can be thought as an α - cyclic decomposed form. Detailed insight into the structure of this decomposition can be obtained from the basic relations (10a)-(10c) and, discussion of condition (14b) is given in Appendix. The main conclusion that follow from the above results is that cyclostationary signal $x(t)$ is not only characterized by the periodicity of its autocorrelation function but, can be also characterized by the periodicity of the mean of its instantaneous frequency.

III. HIGHER-ORDER MOMENTS

Starting from the general relation (7) it is easy to verify

$$\frac{d^n}{d\tau^n} R_x(t,\tau)\Big|_{\tau=0} = (j2\pi)^n \int_{-\infty}^{+\infty} f^n S_x(t,f)df \tag{15}$$

Here we have assumed that $R_x(t,\tau)$ is n-time differentiable function of time lag τ. Following the same procedure as in the case of (10) we can derive

$$E\{f_i^n(t)\} = \frac{1}{(j2\pi)^n}\,\frac{\frac{d^n}{d\tau^n}R_x(t,\tau)\big|_{\tau=0}}{R_x(t,0)} = \frac{\int_{-\infty}^{+\infty} f^n S_x(t,f)df}{\int_{-\infty}^{+\infty} S_x(t,f)df} \tag{16}$$

which is a generally valid expression applicable to any nonstationary random signal. For cyclostationary signal this relation can be further modified by use of definition (7)

$$E\{f_i^n(t)\} = \frac{1}{(j2\pi)^n}\,\frac{\sum_{\alpha}\frac{d^n}{d\tau^n}R_x^{\alpha}(\tau)\big|_{\tau=0}\, e^{j2\pi\alpha t}}{\sum_{\alpha} R_x^{\alpha}(0)\, e^{j2\pi\alpha t}} = \frac{\sum_{\alpha}\int_{-\infty}^{+\infty} f^n S_x^{\alpha}(f)\, e^{j2\pi\alpha t}df}{\sum_{\alpha}\int_{-\infty}^{+\infty} S_x^{\alpha}(f)\, e^{j2\pi\alpha t}df} \tag{17}$$

or with change in notation

$$E\{f_i^n(t)\} = \sum_{\alpha} f_{in}^{\alpha}\, e^{j2\pi\alpha t} \tag{18}$$

Thus we have proved that conclusion made in Section II about (14) can be also extended to higher-order moments of instantaneous frequency and as so is the general property of almost periodic cyclostationary signal.

IV. EXAMPLES

Consider the cyclostationary PM sine wave $s(t) = \cos[\omega_o t + \phi(t)]\ a(t)$, $\phi(t) \in \mathbb{R}$ and the corresponding analytic signal form

$$x(t) = e^{j[\omega_o t+\phi(t)]} \tag{19}$$

for which the periodic autocorrelation function $R_x(t,\tau)$ is defined by

$$R_x(t,\tau) = E\{e^{j[\phi(t)-\phi(t-\tau)]}\}\ e^{j\omega_o\tau} \tag{20}$$

Expression for cyclic autocorrelation function can be evaluated under following two assumptions:

Case #1 ($\phi(t)$ contains no periodicity)

By this we mean that $\phi(t)$ is stationary (or nonstationary) component which implies $R_x^{\alpha}(\tau)$ is non-zero only for $\alpha=0$, i.e.

$$R_x^{\alpha}(\tau) = \begin{cases} \Phi_{\tau}(1,-1)\ e^{j\omega_o\tau}, & \alpha=0 \\ 0, & \alpha \neq 0 \end{cases} \tag{21}$$

where $\Phi_{\tau}(.)$ is time-averaged joint characteristic function for $\phi(t)$ and $\phi(t-\tau)$ respectively, i.e.

$$\Phi_{\tau}(u,v) = \lim_{Z\to+\infty} \frac{1}{Z}\int_{-Z/2}^{+Z/2} E\{e^{j[\phi(t)u+\phi(t-\tau)v]}\}dt \tag{22}$$

Substituting (21) into (10a) we easily obtain

$$E\{f_i(t)\} = f_o$$

Case #2 ($\phi(t)$ contains periodicity)

Repeating the same procedure as in Case #1, cyclic autocorrelation function $R_x^{\alpha}(\tau)$ can be evaluated as

$$R_x^{\alpha}(\tau) = R_{\phi}^{\alpha}(\tau)e^{j\omega_o\tau} \tag{23}$$

and the mean $E\{f_i(t)\}$ can be expressed as

$$E\{f_i(t)\} = f_o + \frac{1}{j2\pi}\,\frac{\sum_{\alpha}\frac{d}{d\tau}R_{\phi}^{\alpha}(\tau)\big|_{\tau=0}\, e^{j2\pi\alpha t}}{\sum_{\alpha} R_{\phi}^{\alpha}(0)e^{j2\pi\alpha t}} = f_o + E\{f_{\phi}(t)\} \tag{24}$$

In order to determine particular $E\{f_{\phi}(t)\}$ we take into consideration the following simple example. Suppose that $R_{\phi}^{\alpha}(\tau)$ is given by

$$R_{\phi}^{\alpha}(\tau) = \begin{cases} a_o, & \alpha=0,\ a_o \in \mathbb{R} \\ a_1\tau, & \alpha=f_1,\ a_1 \in \mathbb{C} \\ -a_1^*\tau, & \alpha=-f_1, \\ 0, & \alpha\neq 0,\ |\alpha| \neq f_1 \end{cases} \tag{25}$$

Substituting (25) for $R_{\phi}^{\alpha}(\tau)$ in (24) it immediately follows

$$E(f_i(t)) = f_o + \frac{a_1}{j2\pi a_o} e^{j2\pi f_1 t} - \frac{a^*_1}{j2\pi a_o} e^{-j2\pi f_1 t} \tag{26}$$

hence periodicity of (26) is easily verified.

APPENDIX

To prove (8) we start with the following identity [12]:

$$E(Im(x'(t)x^*(t))= \frac{1}{2j}[R_{x'x}(t,t) - R_{x'x}(t,t)^*] \tag{A.1}$$

$$R_{x'x}(t,t) = \frac{\partial}{\partial t_1} R_x(t_1,t_2)\Big|_{t_1=t_2=t} \tag{A.2}$$

By use of modified definition (7) for $R_{xx}(t_1,t_2)$ expression (A.2) can be further evaluated as

$$R_{x'x}(t,t) = \sum_\alpha \left[\frac{d}{d\tau}R_x^\alpha(\tau)\Big|_{\tau=0} + j2\pi\alpha R_x^\alpha(0)\right] e^{j2\pi\alpha t} \tag{A.3}$$

which substituted into (A.1) and after subtraction of its complex conjugate counterpart produces the desired relation

$$E(Im(x'(t)x^*(t))= \frac{1}{j}\sum_\alpha \frac{d}{d\tau}R_x^\alpha(\tau)\Big|_{\tau=0} e^{j2\pi\alpha t} \tag{A.4}$$

where we have used the following two identities:

$$R_x(t,0) = \sum_\alpha R_x^\alpha(0)e^{j2\pi\alpha t} = \sum_\alpha R_x^\alpha(0)^* e^{-j2\pi\alpha t} \tag{A.5}$$

$$R_x^\alpha(\tau)^* = R_x^{-\alpha}(-\tau) \tag{A.6}$$

that hold for complex-valued cyclostationary signal $x(t)$. The rest two terms in (8) can be directly obtained in an easy manner by use of (A.4) and the corresponding definitions given in (7).

■

Considering (14b) we note that condition

$$\frac{d}{d\tau} R_x^\alpha(\tau)\Big|_{\tau=0} \equiv 0 \tag{A.7}$$

is met, (excluding trivial case: $R_x^\alpha(\tau)\equiv 0$), if both $Re(R_x^\alpha(\tau))$ and $Im(R_x^\alpha(\tau))$ are even continuous functions of time-lag τ, for every α. But, symmetry property (A.6) which holds for a complex-valued process $x(t)$ implies

$$Re(R_x^\alpha(-\tau)) = Re(R_x^{-\alpha}(\tau))$$

$$Im(R_x^\alpha(-\tau)) = - Im(R_x^{-\alpha}(\tau)) \tag{A.8}$$

so that at least $Im(.)$ can not be even function of τ and thus (A.7) can not hold in the general case.

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