MEDIAN FILTER : FREQUENCY DOMAIN ANALYSIS AND MEDIAN TYPE AVERAGING FILTER

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RÉSUME

SUMMARY

Frequency domain analysis has been presented for a class of signals. It is shown that the median filter acts as a spectrum subtractor. Certain interesting properties of the DFT ratios of input and output of the median filter are given.
For image processing applications it is shown that a modification to the structure of the running median filter, results in a better performance like noise removal, fast convergence and edge preservation. The performance of the average type filter is shown to be useful for these applications.

Introduction -

Median filtering is a simple digital non-linear signal smoothing operation in which median of the samples in a sliding window replaces the sample at the middle of the window. The resulting sequence tends to follow polynomial trends in the original sample sequence. Median filter preserves signal edges while filtering out impulses. It has been difficult to perform frequency domain analysis of the median filter (MF). Tyan [1], Justusson [2], have attempted this but without the universality available to linear filters. This paper presents an analysis of MF in the frequency domain by first considering the ratios of the DFT coefficients of the input and output sequences. Results of this approach for various sequences are interesting.

The large number of comparisons required and the delay introduced by the MF can be minimised by restructuring the running median. In this direction three types of structures have been proposed, viz., Fast Convergence MF (FCMF), Interpolated MF (IMF) and the Median Type Averaging Filter (MTMF). The FCMF median sample occupies the entire window length, while in the IMF the remaining samples in the window of the MF are linearly interpolated. On the other hand the MTMF works on a differential principle and is based on linear operations.

These filters have been implemented for noise removal and edge preservation on 2-D images.

Frequency domain analysis of the MF :

Let $x(n)$ and $y(n)$ be the input and output sequences of the MF respectively. $X(f)$ and $Y(f)$ are the input and output DFT coefficients of the MF. Let $Z(f)$ be the DFT coefficients of the rough part of the input sequence. The interrelationships between X,Y and Z have been analysed for periodic and other deterministic signals.

Let x(n) be a sequence of samples passed through an MF whose window given by 2k+1, k=1,2...etc. Let the output sequence be y(n). If z(n) be the sample set that is not passed by the MF, then :

$$x(n) = y(n) + z(n) \quad ...(1)$$

The sequences in eqn.(1) are all vectors of length equal to the input sequence length, say L. y(n) is called the smooth part nd z(n) is called the rough part of the input sequence. The DFT coefficients of eqn.(1) can be determined and are interrelated :

$$X(f) = Y(f) + Z(f) \quad ...(2)$$

by linearity of the DFT. Rearranging eqn.(2)

$$Y(f) = X(f) - Z(f) \quad ...(3)$$

Thus the MF acts like a spectrum subtractor. However, for general sequences it is possible to determine y(n) and z(n) and their DFT's only after performing the running median. In the case of a few specific signals, interestingly, dividing eqn.(3) by X(f) shows a constant value for every value of f. That is :

$$Y_i(f) / X_i(f) = 1 - Z_i(f)/X_i(f)$$

$$= \text{Const.} \quad ....(4)$$

It can be seen that the simplest case for eqn.(4) to be true, the input x(n) must be periodic. The troughs and peaks are not passed by the MF and their DFT coefficients of Z(f) correspond to such samples. The value of the constant differs from sequence to sequence. z(n) is a function of the window width and the average value of the difference between the maximum/minimum before and after passage through the MF.

Illustrative examples :

Let x(n) be -

x(n) = 0 1 2 1 0 1 2 1 0 1 2 1 0 1 2 1

The out for a window width 3 is :

y(n) 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

which is a step function and so the frequency response is an impulse function. Here the window width is equal to the half period of the input sequence.

Consider another sequence

x(n) = 0 1 2 3 2 1 0 1 2 3 2 1 0 1 2

y(n) = 1 1 2 2 2 1 1 1 2 2 2 1 1 1 2

the rough part z(n) is

z(n) = -1 0 0 1 0 0 -1 0 0 1 0 0 -1 0 0

It can be very easily shown that

$$Y(f) = X(f) - Z(f)$$

The ratio Z(f)/X(f) is equal to 0.5.

Consider another example of a similar periodic sequence

x(n) = 0 1 4 5 4 1 0 1 4 5 4 1 0 1 4 5

then, y(n)

y(n) = 1 1 4 4 1 1 1 4 4 4 1 1 1 4 4

and

z(n) = -1 0 0 1 0 0 -1 0 0 1 0 0 -1 0 0 1

again Z(f)/X(f) is a constant but now it is 0.25.

Consider yet another sequence, with a periodicity longer than the window width.

x(n) = 0 1 2 3 4 5 4 3 2 1 0 1 2 3 4 5

y(n) = 1 1 2 3 4 4 4 3 2 1 1 1 2 3 4 4

z(n) = -1 0 0 0 0 1 0 0 0 0 -1 0 0 0 0 1

The ratio of the DFT coefficients is not constant but is as follows :

1.0, 0.8032, 0.8153, 0.8487, 1.0,
0.5789, 0.6132, 0.6132, 0.6572, 0.0.

For the nine ratios listed a symmetry can be observed around the fifth ratio. The sums of squares of terms symmetrically placed on either side of the above is unity. For arbitrary and unclassifiable signal it is not possible to prespecify the ratio of these coefficients.

Restructuring the Running Median Filter

The FCMF is an improvement over the Ninther proposed by Tukey [3]. All the samples in a window are replaced by the median sample. The inputs are segmented to window length. The outputs therefore are blocks of window length each. The output results in a root sequence. It preserves the edge preservation property of the MF but requires only a single pass. When compared to MF the computation requirement is much smaller.

The I M F :

The input sequence is segmented to window length 2k+1. In each window length of output sequence the central element is replaced by the median sample. The intervening 2k samples between the these median samples are being replaced by interpolation. For a typical application like speckle noise removal linear interpolation can be used.

A comparison of the performances of MF, FCMF and IMF is given in [4]. The computational complexity for FCMF and IMF increases linearly with window width, while for the MF it increases nearly exponential. Performance of the IMF is comparable to that of the running median filter for different kinds of noise. The FCMF preserves all the properties of the MF but its noise performance is slightly inferior.

The MTMF :

In the Median Type Mean Filter the difference between the sample mean and the window mean is used as the filtered output. Consider a finite sequence of length L. Let its mean value be m. Let the window be 2k+1 wide. Let the average value of these 2k+1 samples be w. The difference between the two, ie., m - w replaces the central element of the window and is the output. The window slides from o to L-1 of the samples as in running median operation. The k end samples are appended similar to running MF.

The MTMF ehibits interesting properties. Consider its response to a constant input. In this case the global mean, the median and the window mean are all equal. The output is therefore zero. This behaviour is similar to the MF. Next consider a sequence $\dot{x}(n)$ of length L. The global mean of this sequence is (1/L) x(i). The window means are -

$$\underline{X}(j) = (1/2k+1) \quad x(i)$$

The median on the other hand is obtained by ordering the samples in the window and choosing the middlest of these ordered samples. Neccessarily therefore, the median is less than or= k samples and greater than or= k samples. Thus it can be a first order approximation to the window mean. The MTMF output samples are the difference between X(av) the global mean and $\underline{X}$ the window mean. If the median is a first order approximation to the mean, then then the output sequence of the MTMF will be a first order approximation to the modified MF, where the modification means that the difference between the median and the global mean is passed to the output. Consider now a monotonic sequence. The simplest of cases is where the signal is monotonically rising with a constant difference. The MF and the MTMF both pass the complete sequence. If the difference between successive samples is not constant also both pass the sequence. The MF output is the original sequence, whereas since the difference between the global mean and the window mean for the earlier samples of the sequence is larger the MTMF produces an output sequence which is gradually decreasing. Similarly if the input sequence is a decreasing monotonic, the first window mean is likely to be larger than the global mean whereas the last window mean will be much smaller than the global mean. In terms of order statistics applicable to median filters, this behaviour corresponds to order reversing operation since an increasing monotonic is outputted as a decreasing monotonic and vice-versa. It should be noted that the MMTF behaves as if the input signal to a MF has been time reversed and fed with a scaling operation on the samples. It would be interesting to find the output of the MTMF for an oscillatory input. Depending on the periodicity of oscillation the MTMF will produce an oscillatory output of the same period. However, the amplitude of oscillation is different. This arises because, the MTMF removes any constant or DC bias from the signal.

The MF is very useful for edge detection. Cosider a sample sequence which represents an image. A vertical edge would be represented by a sudden change in the intensity level at one pixel position in all the rows. The left side shows a different intensity level to the right side of this pixel. In MF the edge can be detected by seperable operation of the MF in vertical and horizontal directions. The FCMF and the IMF have poorer edge performance than the MF. The MTMF behaves differently. Because of the difference between global and window means, at an edge where the elements of windows on either side are bound to differ, the output samples also differ. Thus an edge is clearly demarcated in the output sample sequence. But the output of the MTMF contains two sample points representing the edge. The earlier point indicates the end of one region and the second the start of the next region of different intensity. If the true difference between the successive samples at an edge are observed it will be greater than the similar difference between samples at an edge in the MF. This difference can be further enhanced by first performing stretch operation on the image histogram, whereby the global mean value can be increased and the edge difference rises accordingly.

The MTMF is basically a smoother. Thus its noise performance is as good as the MF, the FCMF and the IMF. However, noise spikes are not entirely suppressed. Instead the spike is trimmed and spread over one window length of samples. This can be seen easily. Consider a spike of amplitude 3 times the largest sample in a window. The difference between the global and the window means undergoes a chnage whenever the window contains a spike. This spike however, will be of much smaller magnitude than the valus 3 times the largest sample at the input. But the window means of the the 2k+1 windows undergo the sme effect. Therefore the output samples will show either a positive or a negative window length edge whenever a spike is encountered. Remedial action can easily be taken to correct this. Near flat 2k+1 long sample sequences can easily be identified and extrapolating the average values of window length smples on either side of the region the spike can be removed.

A few examples illustrating the behaviour of the MTMF are shown in Fig.1. It should be pointed out that fom amongst the three types of filters described here, viz., MF,FCMF and the MTMF, only the MTMF retains some information regarding the samples available in the window. The IMF generates additional information in the form of interpolating sample values.

Conclusion

For a class of signals it is possible to define and determine the frequency response of a median filter. The MF appears to be spectrum subtractor. For random signals only the general structure of the frequency sponse can be described. Modifications to the MF lead to other useful filter structures. The FCMF for example minimises the delay and number of multiplications while retaining most of the properties of the MF. The IMF improves on both the computational and smoothing aspects. The MTMF is a structurs similarb to the MF operationally. Though the inputs are not ordered for processing, surprisingly it behaves like an order reversing filter. The MTMF has many properties similar to the MF. DC blocking, passing of root sequences, of course after order reversal, edge preservation, as also spiky noise redction. In view of the simplicity of implementation and the reasonable good performance the MTMF can be used for image processing.

References

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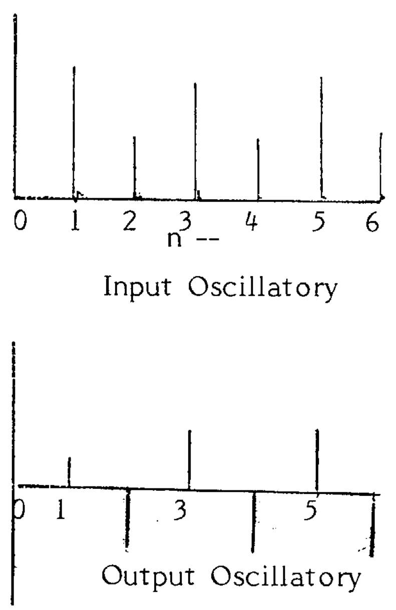

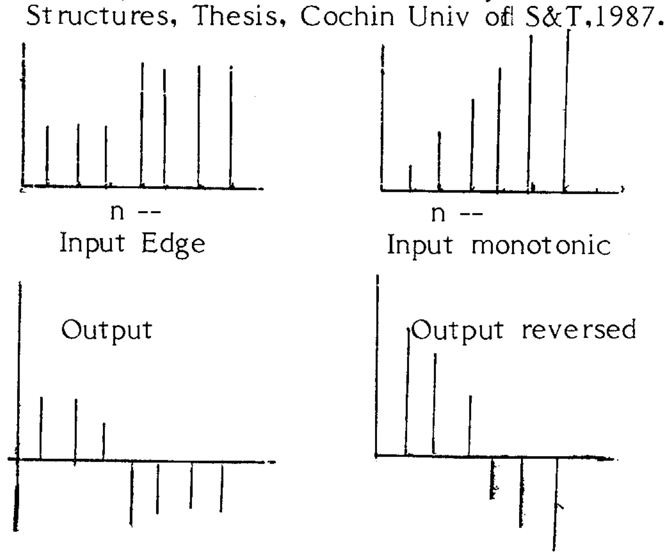

Fig. Typical behaviour of the MTMF.