On the Time Delay Measurements by the Average Magnitude Difference Function

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RESUME

La qualité de l'estimation du retard par la Fonction Moyenne des Différences Absolues (AMDF) est presque égal à celle fournie par la méthode à inter-corrélation, avec une structure computationelle plus simple. Souvent on ne tient pas suffisamment compte des problèmes du débit d'échantillonage nécessaire pour obtenir l'estimation AMDF.

Dans cet article on traite le problème d'interpolation, et on donne une expression simple de la variance finale d'estimation. On fournit aussi la variance du retard estimé par la méthode à inter-corrélation dans les mêmes conditions.

SUMMARY

It is known that the accuracy of the Average Magnitude Difference Function (AMDF) based time delay estimate approaches the one of the classical cross-correlation-based estimate with a lower computational effort. However, some of aspects concerning the sampling rate necessary to search the extremum of the AMDF are often negliged.

In this work, the interpolation problems are addressed and the theoretical values of the variance of the estimate are given for a reference case. For comparison purposes, the theoretical accuracy is also given under the same conditions for the cross-correlation based (Direct) estimate.

1 Introduction

A special attention is currently devoted to the time of delay estimation (TDE) techniques. One of the most typical application is the direction of arrival and range estimation in multisensor arrays. Often, operative conditions require both accurate and computationally simple algorythms. Optimum TDE consist in cross-correlating signals after a suitable pre-filtering. An excellent review of these techniques along with a discussion about the Rao-Cramer bounds is given in [1].

Recently, an empirical comparison of the cross-correlation TDE estimates with sum-based algorythms has been performed [2]. It has been observed that the so-called Average Magnitude Difference Function (AMDF) yields very good performance while requiring only sums. As a matter of fact, the AMDF is employed for pitch measurements in speech analysis. An euristic discussion of the performance of the AMDF in such applications was reported in [3]. However, no special mention is made in the referred works to the sampling requirements related to the use of the AMDF. In this contribution, we consider in detail the sampling and interpolation procedures for reaching the minimum of the AMDF, the abscissa of which is the searched delay time. Then, we give explicit theoretical values of the variance of the AMDF estimate, and compare them to the Rao-Cramer bounds. A check of these results is provided by Montecarlo trials.

2 The TDE problem

We refer to a scheme where a signal $s(t)$ and its delayed version $s(t-D)$ are received by a pair of sensors, which outputs are respectively

$$x_1(t) = s(t) + n_1(t) \qquad (1)$$

$$x_2(t) = s(t-D)+n_2(t)$$

The signal $n_1(t)$ and $n_2(t)$ represent independent Gaussian and white measurements noises.

The TDE problem consists in estimating the parameter D from the pair of the measured signals. Using the classical Direct Correlator (DC) we search the time τ which maximizes the sample cross-correlation product

$$\hat{R}_{DC}(\tau) \doteq \frac{1}{N}\sum_{k=1}^{N} x_1(kT)x_2(kT+\tau) \qquad (2)$$

where T is the sampling interval, and NT is the estimation window. Ideally, the maximum occurs for $\tau = D$. In practice we have $\tau = \hat{D}_{DC} \neq D$.

Estimation errors may be caused not only by the measurement noise, but also by the finite extension of the window, NT.

In principle, the TDE procedure entails the evaluation of $\hat{R}_{DC}(\tau)$ for an infinite number of points. In practice, $x_2(t)$ is sampled every T seconds, so that τ can assume only the values nT (n integer). When the time T is comparable with the desired resolution, it is sufficient to find the value n_m of n corresponding to the maximum cross-correlation, and no further action is required. Very often this is not the case, and an interpolation is necessary to perform the desired TDE estimate. Assuming that the signal $x_2(t)$ is band-limited into the range $\pm\frac{1}{2T}$, $x_2(t+\tau)$ can be calculated using either zero padding or linear phase addition with the FFT technique.

A more simple technique consists in interpolating $\hat{R}_{DC}(\tau)$ from its samples because it is band-limited in the same range as $x_2(t)$. FFT-based techniques can be employed as well. However, let us observe that $\hat{R}_{DC}(\tau)$ presents the special feature of having a pronounced main lobe around $\tau = D$ with an (ideally) symmetric shape. Due to the finite bandwidth, it can be expressed in Taylor series around D, and approximated up to the second order by a convex parabola.

$$\hat{R}_{DC}(\tau) = a\tau^2 + b\tau + c \qquad (3)$$

This approximation is reasonable if the time τ is small enough with respect to the widht of the main lobe.

Using the approximation (3), the TDE procedure can be divided in two steps:

- Location of the absolute maximum sample $\hat{R}_{DC}(n_mT)$ and estimation of two samples within the main lobe, $\hat{R}_{DC}(n_mT+\theta)$, $\hat{R}_{DC}(n_mT-\theta)$;
- Evaluation of the apex of the parabolic model as final TDE

$$\hat{D}_{DC} = -\frac{b}{2a} = -\frac{\theta}{2}\cdot \frac{\hat{R}_{DC}(n_mT+\theta)-\hat{R}_{DC}(n_mT-\theta)}{\hat{R}_{DC}(n_mT+\theta)-2\hat{R}_{DC}(n_mT)+\hat{R}_{DC}(n_mT-\theta)} \qquad (4)$$

The samples at $n_mT \pm \theta$ are directly available if $\theta = T$ (in this case an oversampling factor ≥ 2 with respect to the Nyquist rate is required).

Otherwise, they can be calculated by means of an ideal interpolation either in the τ domain or using the FFT.

The accuracy of this estimate can be measured by calculating its variance

$$\mathrm{Var}\left\{\hat{D}\right\} = E\left\{\left(\hat{D}-D\right)^2\right\} \qquad (5)$$

In our context we assume that the SNR is sufficiently *high* so that the probability of finding a false absolute maximum into sidelobes is negligible.

The general expression of the variance in this case is omitted here for the sake of compactness, and is reported in [5].

Here let us consider the particular case when the pairs of samples employed in the estimate of the cross-correlation (2) are reciprocally uncorrelated. This occurs when T is very large with respect to the duration the acf of $s(t)$, but also corresponds to the particular case of white signal embedded in white noises and very small θ.

In this case we obtain, for Gaussian signals

$$\mathrm{Var}\{\hat{D}_{DC}\} = \frac{1}{N}\frac{3}{4\pi^2B^2}\left(\frac{1+2\mathrm{SNR}}{\mathrm{SNR}^2}+1\right) \qquad (6)$$

where SNR is the signal to noise ratio, assumed equal for both the channels, and B is the half bandwidth of the signals.

It is interesting to compare this value with the Rao-Cramer bound for flat spectrum given in [1].

$$\sigma^2_{RCLB} = \frac{1}{N}\frac{3}{4\pi^2B^2}\left(\frac{1+2\mathrm{SNR}}{\mathrm{SNR}^2}\right) \qquad (7)$$

It is evident that the Direct accuracy of the cross-correlation technique does not increase without limits when the measurements noise vanishes, and does not approach the behavior of a Maximum Likelyhood estimator.

In fig.1 the variance of the Direct estimator and of the RCL bound versus SNR is reported in logarythmic scales, along with some simulation results (3000 runs).

3 The AMDF estimate

The Average Magnitude Difference Function is defined as

$$\hat{R}_{AM}(\tau) \doteq \frac{1}{N}\sum_{k=1}^{N}|x_1(kT) - x_2(kT+\tau)| \quad (8)$$

The AMDF TDE technique consists in searching the time $\hat{D}_{AM}$ for which the AMDF attains its minimum. The main advantage of this method consist in the use of additions only. On the other hand, the methods for interpolating the cross-correlation does not longer apply to the AMDF, because this is not band limited. In particular, for Gaussian processes, we have

$$E\left\{\hat{R}_{AM}(\tau)\right\} = \frac{\pi}{2}\sqrt{2\left[R_s(0) - R_s(\tau - D)\right] + \sigma_{n_1}^2 + \sigma_{n_2}^2} \quad (9)$$

where $R_s(\tau)$ is the acf of the signal and $\sigma_{n_1}^2$, $\sigma_{n_2}^2$ are the variances for the noise. The shape of the AMDF fits a concave hyperbola around $\tau = D$, which cannot be interpolated using the *sinc* function or a parabola, unless high sampling rate is employed.

However this fact suggests to square the AMDF in order to locate its minimum with a parabolic interpolation. Thus we may devise to employ the following TDE estimate, for Gaussian processes

$$\hat{D}_{AM} = -\frac{\theta}{2}\cdot \frac{\hat{R}_{AM}^2(n_mT+\theta) - \hat{R}_{AM}^2(n_mT-\theta)}{\hat{R}_{AM}^2(n_mT+\theta) - 2\hat{R}_{AM}^2(n_mT) + \hat{R}_{AM}^2(n_mT-\theta)} \quad (10)$$

where n_mT is the position of the absolute smallest sampled AMDF value and the times $n_mT \pm \theta$ are supposed to lie into the main lobe.

Let us consider the accuracy of this estimate. Proceeding as in the case of the Direct estimate, we can calculate the approximate variance of the AMDF estimate by assuming that the probability of ambigous peak detection is negligible.

Expanding in series equation (10) around the time delay D and retaining terms up to the second order we calculate both the bias and the variance. For this purpose, the moments of the AMDF given in [4] are employed.

The details and the general results of this analysis are reported in [5]. For the particular case of white signals in white noises and very small values of θ we have, for Gaussian processes

$$\mathrm{Var}\{\hat{D}_{AM}\} = \frac{\pi}{2}\frac{1}{N}\frac{3}{4\pi^2B^2}\frac{2+2\mathrm{SNR}}{\mathrm{SNR}^2} \quad (11)$$

Apart from the factor $\frac{\pi}{2}$, the main difference with the Direct estimate is the fact that the variance tends to zero for increasing SNR. This is visible in fig.2 where (10) is plotted versus SNR along with simulation results (3000 runs).

These curves justify the empirical data reported in [2].

On the same diagram, the variance of the *Average Square Difference Function* (ASDF) is reported for comparison.

The ASDF is defined as

$$\hat{R}_{AS}(\tau) \doteq \frac{1}{N}\sum_{k=1}^{N}\left[x_1(kT) - x_2(kT+\tau)\right]^2 \quad (12)$$

The ASDF TDE consists in searching the value of τ for which the absolute minimum occurs. The ASDF requires computation of squares and sums. It is computationally more simple then the Direct estimate. Moreover it is, almost approximately, band limited and can be interpolated using the same methods employed for the cross-correlation. Thus, we consider the estimate

$$\hat{D}_{AS} = -\frac{\theta}{2}\cdot \frac{\hat{R}_{AS}(n_mT+\theta) - \hat{R}_{AS}(n_mT-\theta)}{\hat{R}_{AS}(n_mT+\theta) - 2\hat{R}_{AS}(n_mT) + \hat{R}_{AS}(n_mT-\theta)} \quad (13)$$

with the same assumption made for the AMDF estimate.

Evaluation of the variance of this estimate for Gaussian white signals in white noises for very small θ gives

$$\mathrm{Var}\{\hat{D}_{AS}\} = \frac{1}{N}\frac{3}{4\pi^2B^2}\frac{2+2\mathrm{SNR}}{\mathrm{SNR}^2} \quad (14)$$

This is a scaled value of the variance of the AMDF estimate which approaches an efficient behavior for high SNR. This is visible in fig.2.

4 Conclusion

The AMDF technique for estimating the time of delay between two signals is a very efficient one because it requires only sums and exhibit high accuracy. We have shown that it can be implemented at the same low sampling rate as the cross-correlation estimate, using an appropriate non-linear interpolation, without degrading its performance.

References

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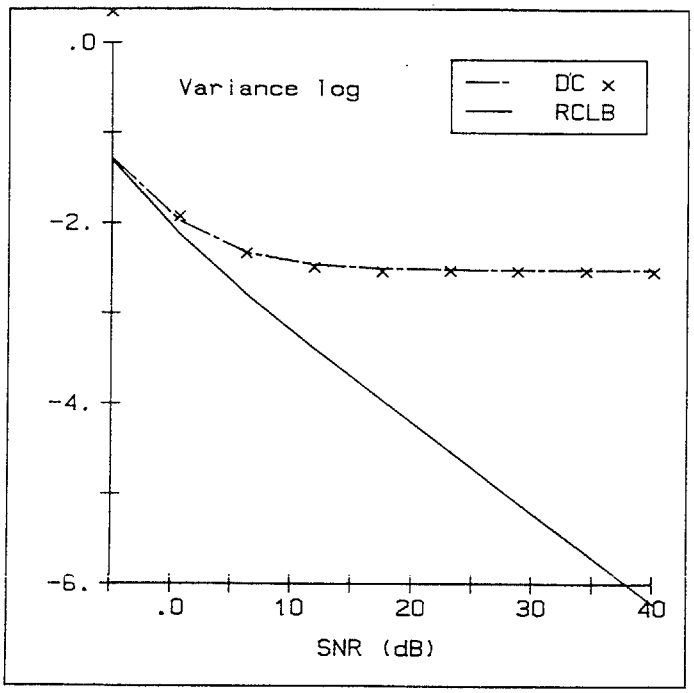

Figure 1: *Variance of the DC TDE versus SNR compared to the Rao-Cramer Lower bound. The symbols x indicate simulation results (3000 runs).*

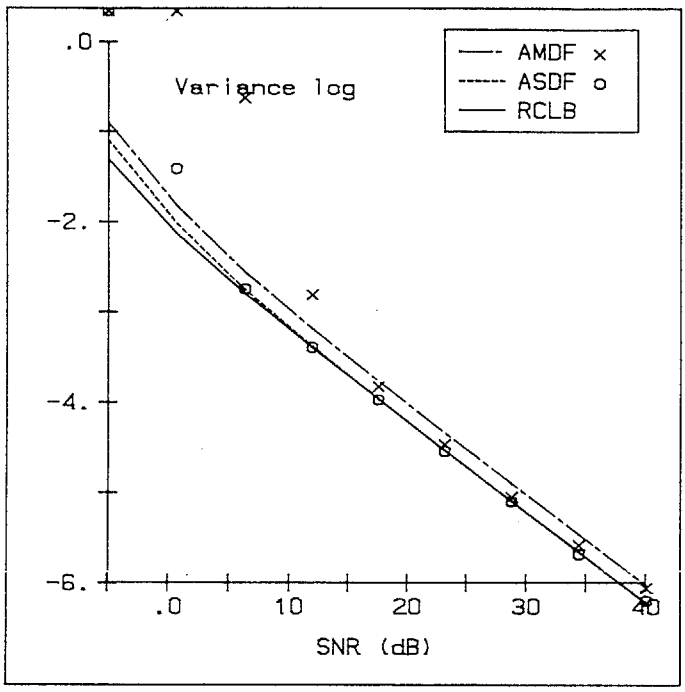

Figure 2: *Variance of the AMDF TDE and of the ASDF TDE versus SNR compared to the Rao-Cramer Lower bound. The symbols x and o indicate simulation results (3000 runs).*

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