SIGNAL PROCESSING IN THE MIXED TIME-FREQUENCY DOMAIN USING SVD AND WIGNER-VILLE TRANSFORM TECHNIQUES

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RESUME

La distribution de Wigner-Ville (DWV) est une représentation bidimensionnelle temps-fréquence d'un monodimensionnel signal temporel et il est évident qu'il y a une redondance de l'information dans cette distribution. Dans l'article on présente le problème de l'utilisation optimale de la décomposition selon les valeurs singulières (DVS) pour la compression du spectre de Wigner-Ville. On a formulé les conclusions générales concernants ce problème. On a présenté des examples du traitement des signaux dans le plan temps -fréquence où on utilise la DWV ainsi que la DVS. On a comparé l'efficacitée de l'utilisation de la méthode du traitement temps-fréquence du signal, présentée ci-dessus, avec l'efficacité de certaines autres méthodes utilisées pour la filtration du bruit.

SUMMARY

The Wigner-Ville distribution (WVD) is a two dimensional time-frequency (TF) representation of a one dimensional time signal and it is obvious that there exists a redundancy of information contained in it. The paper discusses a problem of optimal application of the singular value decomposition (SVD) for the compression of the WV-spectrum. General conclusions are formulated in it and some examples of signal processing in the mixed TF domain by means of the WVD and the SVD are given. A short performance comparison is made between the presented technique and the other methods of signal processing in the case of noise filtering.

1. INTRODUCTION

The Wigner-Ville distribution (WVD) [1-4] is one of the mixed time-frequency signal representations (MTFRs). Its main advantages over the other MTFRs from the Cohen's class (short-time Fourier transform, Richaczek distribution, Page and Levin representations, etc.) are better resolution (higher energy concentration in the mixed time-frequency (TF) plane) and direct viewing and simple computing of frequency modulation law for monocomponent signals. The cost one must pay for good resolution of the WV-analysis are negative values and so-called cross-terms that have no physical significance but can appear in the resultant spectrum [5]. The cross-terms generation is caused by the bilinearity of the WV-transform.

On the other hand the WVD is a two dimensional time-frequency representation of a one dimensional time signal (usually contaminated with noise) and there exists a redundancy of information contained in it.

The problem is how to eliminate the negative values, the cross-terms, the information redundancy and noise component from the signal WV-spectrum without loosing its good resolution. Appropriate smoothing of the WVD can be tolerated in many applications but it is not an optimal solution.

The singular value decomposition (SVD) is like the WVD also a high resolution tool and it is widely used for compression of 2D data [6-9]. The reduction of information redundancy is realized in it by means of orthogonal decomposition of the matrix and not by its 2D smoothing. The application of the SVD for the compression of the WVD was proposed by Marinovic and Eichmann in 1985 [10]. They suggested a direct decomposition of the TF matrix of the WV-spectrum. A new approach relying on the SVD of the WV-kernel was proposed by Boashash and Whitehouse [11].

The present paper is the continuation and extension of [10] and the authors would like to answer the following questions:
1. For what kind of signals the application of the SVD for the compression of the WV-spectrum is optimal?
2. Is it possible to use the SVD for elimination or reduction of the cross-terms in the WV-spectrum?
3. What is the efficiency of time-frequency signal processing [12-14] by a joint application of the SVD and the WVD?

2. OUTER PRODUCT INTERPRETATION OF THE WVD

In the discrete case the smoothed (M>1) cross pseudo-WVD (CWD) is given by the following formulae [2]

$$CWD^{2M-1}_{2N-1}(n,\theta)=2\sum_{k=-N+1}^{N-1} Z(n,k)e^{-j2\theta k} , \qquad (1)$$

$$Z(n,k)=|h_N(k)|^2 \sum_{m=-M+1}^{M-1} g_M(m)X(n+m+k)Y^*(n+m-k), \qquad (2)$$

where $X(i)$, $Y(i)$ denote complex-value discrete-time analytic signals associated with real-value discrete time signals $x(i)$, $y(i)$ while $h_N(k)$ and $g_M(m)$ are symmetric, normed data windows with nonzero values 2N-1 and 2M-1, respectively ("*"- complex conjugation, "n"- discrete time, "θ"- discrete frequency).
For M=1 the WV-kernel $Z(n,k)$ is reduced to the form

$$Z(n,k)=|h_N(k)|^2X(n+k)Y^*(n-k) \qquad (3)$$

and (1)(3) defines unsmoothed CWD. When $h_N(k)$ is a rectangular window, (3) can be reduced even further

$$Z(n,k)=X(n+k)Y^*(n-k) . \qquad (4)$$

When $X(i)=Y(i)$, (1)(2), (1)(3) and (1)(4) define appropriate auto WVD (AWD).

Let assume that M=1 and $h_N(k)$ is a rectangular window and the cross pseudo-WVD is defined by equations (1)(4). Let variables in (4) be changed in the following way: $n+k=p_1$, $n-k=p_2$ [15]. In this case $Z(n,k)$ can be interpreted as an outer product of two vectors: $X(p_1)$ and $Y^*(p_2)$, and the TF matrix of the CWD defined by (1)(4) can be shown as a result of the Fourier transform performed on 2N-1 data $X(p_1)Y^*(p_2)$ lying on the lines perpendicular to the axis "n", symmetrically to these pairs $X(p_1)Y^*(p_2)$ which belong to that axis. This fact is demonstrated in figure 1. It shows seven segments connecting elements that are necessary for computing the CWD (N=4, 2N-1=7) for time moments n=4,5,6,.,10. It is important to note that not all the elements lying on lines perpendicular to the axis "n" can be used for the computation of the WV-spectrum. The used ones are marked with cross in figure 1.

Figure 1 gives interesting insight into the computational structure of the unsmoothed CWD (1)(4). It is easy to expand this approach a step further and to interpret a computation of the unsmoothed CWD with window $h_N(k)$ different from rectangular (1)(3) and of the smoothed CWD (1)(2) in the outer product plane. In the first case we only multiply elements $X(n+k)Y^*(n-k)$ of the kernel (4), n=4,5,6,.,10, k=-3,.,0,.,3, lying on the marked lines in fig.1 with appropriate coefficients $|h_N(k)|^2$. In the second case we first make summations of 2M-1 pairs $X(n+k)Y^*(n-k)$ lying on neighbouring lines (k=const, $n=n_o+m$, m=-M+1,.,0,.,M-1; n_o-time point for which the smoothed spectrum is computed; in fig.1 n_o=4,5,6,.,10) with weights $g_M(m)$ and then we multiply the resultant smoothed kernel with coefficients $|h_N(k)|^2$ as it was described above.
Fig.1 can also be used for the analysis and development of the pipeline and parallel algorithms for on-line effective computation of the WV-spectrum [16,17] as well as it can help the understanding of the signal synthesis procedures [13-14] from the modified AWD and CWD.

3. SINGULAR VALUE DECOMPOSITION OF THE WV-SPECTRUM

3.1 Theoretical background

Let $A\in\mathbb{C}^{m\times n}$ denote a complex matrix with rank r ($r<\min(m,n)$). Then unitary matrices $U\in\mathbb{C}^{m\times m}$ and $V\in\mathbb{C}^{n\times n}$ exist such that [9,10]

$$A=UDV^H=\sum_{i=1}^{r}\sigma_i\bar{u}_i\bar{u}_i^H=\sum_{i=1}^{r}A_i \qquad (5)$$

where

$$D=\begin{bmatrix} S & 0\\ 0 & 0\end{bmatrix}$$

and $S=\mathrm{diag}(\sigma_1,\sigma_2,\sigma_3,.,\sigma_r)$, $\sigma_1\geq\sigma_2\geq\sigma_3\geq.\geq\sigma_r\geq 0$ ("H" denotes transposition and conjugation). (5) defines the decomposition of the matrix A on the singular system $(\sigma_i,\bar{u}_i,\bar{v}_i)$, i=1,2,3,.,r. It represent a sum of outer products of vectors $\bar{u}_i$ and $\bar{u}_i^H$ which are taken with weights σ_i called the singular values of matrix A. When only the first "p" singular values σ_i are significant, summation in (5) can be reduced to the first "p" components [10]

$$A^p=\sum_{i=1}^{p}\sigma_i\bar{u}_i\bar{v}_i^H=\sum_{i=1}^{p}A_i , \quad p<r . \qquad (6)$$

The matrix A^p received this way has the rank p and it is the best approximation of the matrix A in the least squares (the Frobenious norm) sense in comparison with the other matrices of this rank. The absolute approximation error ε_a is given by the formula [10]

$$\varepsilon_a=\|A-A^p\|=[\sum_{i=p+1}^{r}\sigma_i^2]^{1/2},$$

where

$$\|A\|=[\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^2]^{1/2}=[\sum_{i=1}^{r}\sigma_i^2]^{1/2}.$$

In the case of the WVD, the SVD matrix approximation (6) can be applied both to the time-time matrix of the WV-kernel (4) (marked in fig.1 with seven segments) and to the time-frequency matrix of the computed WV-spectrum (1)(4). Since the Fourier transform is a unitary transformation and it does not change a singular system of a matrix [10] both methods give the same results. Thus the second method is prefered. It allows to treat the SVD of the WVD as a one of many possibilities of modification of the computed WV-spectrum and it will be discussed further.

The idea of using the SVD for the compression of information contained in the matrix of the WV-spectrum is based on the assumption that signal energy is concentrated in the matrices A_i of the p first singular values (6), while the noise energy is spread more or less uniformly in all matrices A_i, i=1,.,n (fig.1,2,3 in [10]). It is due to the fact that the analyzed signal is expected to have in general a time-varying but well-defined frequency structure as opposed to the noise and it is hoped that this structure has a geometrical pattern easily decomposed by the SVD (easy means with a few singular values) [6]. The use of (6) leads in this situation to the reduction of noise contribution and to the simplification of spectrum interpretation. When it is additionally possible to synthesize a signal from the TF matrix of the WV-spectrum after its modification by means of the SVD, the described technique can be used also for time-frequency noise filtering (sec.4) [12-14].

3.2 Remarks on experimental results

Efficiency of the SVD application for the compression of the WV-spectrum was tested in the computer system described in [16]. The experiments performed result in the formulation of the following remarks [17]:

SIGNAL WITHOUT NOISE

1. Using the SVD for the compression of the WV-spectrum is effective especially in the case of TF matrices with vertical-horizontal structures in which changes of elements values are parallel to the time axes or to the frequency axes [6]. In this case the matrix has only a few significant singular values (spectrum of: one sinusoid -1 sv, two sinusoids -2 sv's, sinusoid with Gaussian time envelope- 1 sv, two sinusoids with Gaussian time envelopes which do not coincide in time -4 sv's; but spectrum of chirp signal with Gaussian time envelope -several sv's).

2. In order to achieve a physically true approximation of the signal spectrum (frequency modulation laws) by means of the SVD it is necessary to make the summation in (5) to the last significant sv of the signal. In general no temporary level of decomposition (A^i, $i<p$) (6) can give sufficient information about the signal spectrum structure, especially when the "outer" cross-terms [5] of the WVD are present. In general the cross-terms components as a whole or as a part do not appear separately in any matrix associated with one or some sv's but they always coincide with a "signal" part of the spectrum. As a result of this fact it is impossible to eliminate the cross-terms from the WV-spectrum by means of the SVD [17]. The process of the decomposition itself can differ significantly for a slightly different spectra.

SIGNAL WITH NOISE

3. When an additive noise W(i) is added to the signal S(i) (X(i)=S(i)+W(i)) the unsmoothed AWD of the analyzed data X(t) can be represented in the following form [1,5]:

$$AWD^{X}(n,\theta)=AWD^{S}(n,\theta)+AWD^{W}(n,\theta)+2Re\{CWD^{S,W}(n,\theta)\} \tag{7}$$

where $AWD^{X}(n,\theta)$, $AWD^{S}(n,\theta)$ and $AWD^{W}(n,\theta)$ denote the auto pseudo-WVDs of X(i), S(i) and W(i) (auto means that X(i)=Y(i) in (1)(4)) and $CWD^{S,W}(n,\theta)$ denotes the cross pseudo-WVD between S(i) and W(i). We are interested in extracting the $AWD^{S}(n,\theta)$ from the $AWD^{X}(n,\theta)$ by means of the SVD. Since the SVD is not an "additive" operation in the sense of singular values σ_i (5), sv's of the matrix $AWD^{X}(n,\theta)$ are not the sum of appropriate sv's of the matrices AWD^{S}, AWD^{W} and $2Re\{CWD^{S,W}\}$, direct comparison of sv's can give only approximate information about participation of the signal, noise and "cross" components in the approximation matrices A_i of $AWD^{X}(n,\theta)$ (6)(7). However, in general using the SVD is purposeful only when the matrix $AWD^{S}(n,\theta)$ has few sv's and they are bigger than appropriate sv's of matrices $AWD^{W}(n,\theta)$ and $2Re\{CWD^{S,W}(n,\theta)\}$ [10]. When the matrix AWD^{S} has not a vertical-horizontal structure this condition most often is not fulfilled [17]. In order to obtain such structure if it is possible, we can rotate the matrix AWD^{X}, make the SVD and rotate the compressed matrix back [17]. This technique enlarges the possibilities of the efficient SVD application to a larger class of signals.

4. PROCESSING OF THE SIGNAL IN THE MIXED TIME-FREQUENCY DOMAIN

Computing the WVD, performing the SVD compression of the WV-spectrum and synthesizing the signal from the modified spectrum is an example of the so-called time-frequency signal processing in which operations performed in the TF domain are crucial [12-14]. Illustrations of this type of processing can be found in [16][17].

Among other things the proposed processing technique can be used for noise filtering. Table 1 illustrates a short performance comparison of the SVD method with other noise filtering methods: bandpass filtering (BPF), use of the smoothed AWD (1)(2) and synthesis of the signal directly from the smoothed spectrum (SMOOTH) [13], multiplication of the WV-spectrum by a mask matrix with elements equal to zero or one, and synthesis of the signal from the modified spectrum (MASK) [14]. SNR coefficient is computed in the way defined in [13]. The SVD method is applied <u>without</u> the additional <u>rotation</u> of the TF matrix. The comparison is made for three different input signals: sinusoid (SINUS), sinusoid with Gaussian time envelope (SINUS+GAUSS), chirp signal with Gaussian time envelope (CHIRP+GAUSS).

5. CONCLUSIONS

The joint application of two high resolution tools: the Wigner-Ville distribution (WVD) and the singular value decomposition (SVD) in the time-frequency (TF) scheme of signal processing can give very good results only when TF matrix of signal spectrum have vertical-horizontal structure. Rotation of TF matrix before the SVD is suggested, provided that it can result in such structure.

Comparison of different signal processing methods has been made and it has turned out that in the case of noise filtering the proposed technique is the most efficient one in the case when the signal spectrum has a required vertical-horizontal form.

Generally it is impossible to reduce or eliminate the cross-terms from the WV-spectrum by means of the SVD.

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Table 1

	SNR_{in} [dB]	SNR_{out} [dB]			
		BPF	SMOOTH	MASK	SVD
SINUS	2.6	33.1	34.2	40.7	37.3
SINUS+GAUSS	2.6	25.2	27.6	28.9	39.0
CHIRP+GAUSS	2.6	16.2	17.8	19.5	16.0

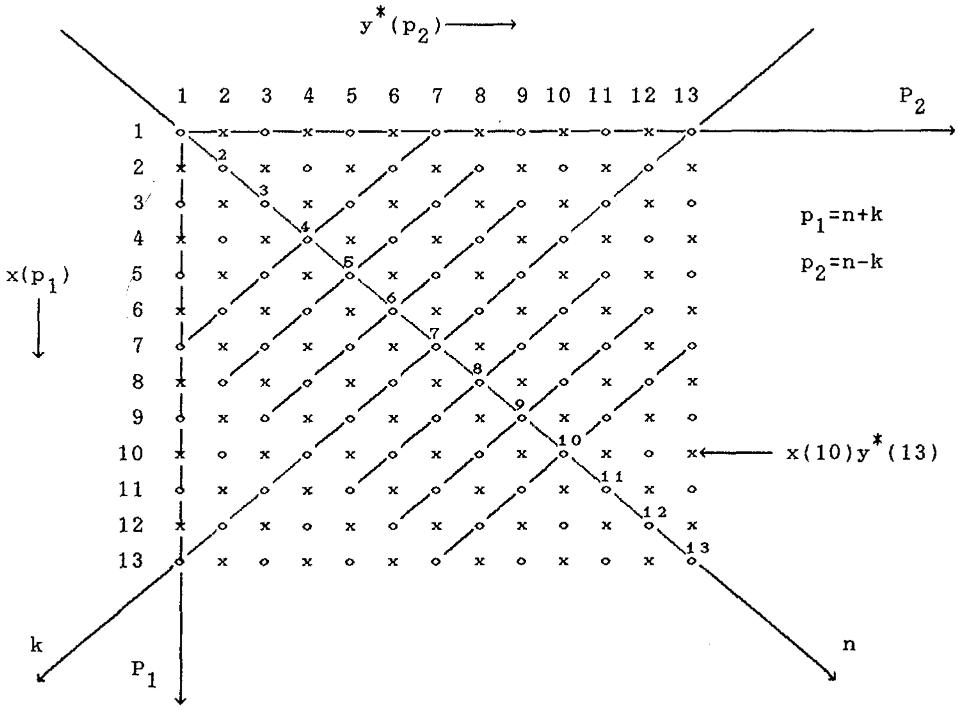

Figure 1