

THE ROLE OF PERIODOGRAM ENVELOPE IN SPECTRAL ESTIMATION

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Le concept d'enveloppe spectrale dérivé du périodogramme donné est introduit. L'intérêt de cette fonction est montré, soit en non-paramétrique soit en paramétrique rapprochement. Les propriétés agréables de l'enveloppe associé avec la magnitude du sur nommé spectrum analytique indiquent la nouvelle fonction comme la candidate la plus appropriée comme restrictions en estimation spectrale variacionelle. Finalement, un estimateur spectrale ARMA bien supporté à partir de restrictions de corrélation et d'enveloppe est dérivé. En éffect, l'usage de l'enveloppe du périodogramme en astimation spectrale, soit pour estimatives finales soit pour prè-estimatives en procédés plus complexes, doit être préféré vis-à-vis d'autres possibles fonctions comme le cepstrum. Ceci se doit au caractère lisse, à la stabilité statistique et au modèle linéaire, dérivé de la fonction d'autocorrélation causale, que l'enveloppe du spectrum ou du periodogramme a.

The concept of spectral envelope derived from the given periodogram is introduced. The interest of this function is shown both in the non-parametric as well as in the parametric approach. The nice properties of the envelope associated to the magnitude of the so-called analytic spectrum, aprize the new function to be the most suitable candidate for additional constraints in variational spectral estimation. Finally, a well supported ARMA spectral estimate from correlation and envelope constraints is derived. In fact the use of periodogram envelope in spectral estimation, both for final estimates or for prior estimates in more complex procedures, should be preferred with respect other possible functions like cepstrum. This is due to the smooth character, the statistical stability and the linear model, derived from the causal autocorrelation function, that the spectral or the periodogram envelope has.

1. INTRODUCTION

Given a signal record there are two basic representations of it, which are derived from its geometric representation in the complex plane. Most of the cases we are concerned with the real and imaginary parts of the signal under analysis but the alternative, the envelope and phase representation have been proved very efficient in many cases and applications. This work deals with the envelope/phase representation of the power spectral density and its interest in variational approaches for spectral estimation.

Further insight can be gained in giving an explanation to the name of spectral envelope methods used to refer parametric methods for spectral estimation. It is well known in communications systems that the product of two signals named $a(t)$ and $c(t)$ has an envelope determined by $a(t)$ and the envelope of $c(t)$.

$$x(t) = a(t).c(t) \quad (1.a)$$

$$e_x(t) = |a(t)|.e_c(t) \quad (1.b)$$

In order to satisfy (1.b) it is required that $a(t)$ is a low pass frequency signal and $c(t)$ is a high or band pass frequency signal which does not overlap $A(\omega)$. Furthermore, if $e_c(t)$ is a constant the previous statements proves that, under some circumstances, the information from one signal embeded in multiplicative noise can be recovered from the envelope of the given signal.

The property mentioned is of capital importance in terms of spectral estimation. Let us assume that a signal model with frequency response $H(h)$ produces $X(\omega)$ when the input is a white noise record with Fourier transform $W(\omega)$. Being the overlap of $\lambda(t)$ and $\omega(t)$ small it can be said that the envelope of $X(\omega)$ becomes the magnitude of $H(\omega)$ multiplied by the envelope of $W(\omega)$. At the periodogram level, it can be said that approximately the periodogram envelope of $X(\omega)$ is equal to the magnitude square of the transfer function $H(\omega)$ multiplied by the periodogram envelope of the white noise input. This last term is a constant over all the frequency band of interest.

$$E_P X^2(\omega) = |H(\omega)|^2 . E_P \omega^2(\omega); E_P \omega^2(\omega) = K_0 \quad (2)$$



Based in (2) it is why, in trying to smooth the periodogram behaviour, the maximum entropy approach produces an estimate of the spectral envelope. This point of view, even it seems to be artificial, supports somehow the similarity of parametric approaches for spectral estimation results with the given data periodogram envelope.

Nevertheless, the way this work is going to focuss the role of periodogram envelope is quite different from it is stated above, the starting point will be motivated by the interest, like in speech analysis/synthesis and recognition, of the periodogram envelope. The potential of periodogram envelope is shown by its use in getting new signal models and a well supported ARMA variational approach.

2. THE PERIODOGRAM ENVELOPE

The content of this section can be viewed as the non-parametric approach of the framework supported by the envelope in spectral estimation problems. Given the periodogram $P(\omega)$, computing its Hilbert transform $H_P(\omega)$ the squared envelope is given by (3).

$$E_P^2(\omega) = P(\omega)^2 + H_P^2(\omega) \quad (3)$$

The interest of $E_P^2(\omega)$ instead $P(\omega)$ in practical applications resides in two distinctive features. The first one is its smooth character which avoid the undesired sidelobes of the analysis window which show up in the periodogram. This smooth behaviour can be viewed as a consequence of removing the instantaneous phase from the periodogram.

$$P(\omega) = E_P(\omega) \cos \phi_P(\omega) \quad (4)$$

Second property of $E_P(\omega)$ is the robustness or statistical stability over different sample data records belonging to the same random process. To support this experimental behaviour of the periodogram envelope, we need to report the concept of causal autocorrelation and analytic spectrum.

The autocorrelation function of a random process $\{x\}$ is an even function which can be decomposed in a causal and non-causal functions as shown in (5).

$$R(z) = R_C(z) + R_C(z^{-1}) \quad (5.a)$$

$$R_C(z) = \frac{r(0)}{2} + \sum_{q=1}^{\infty} r(q) \cdot z^{-q} \quad (5.b)$$

The Fourier transform of the causal acf can be named as the analytic spectrum, $A_P(\omega)$ defined as $R_C(\exp(j\omega))$ [1]. Note that the magnitude of the analytic periodogram is the periodogram envelope.

$$A_P(\omega) = R_C(\exp(j\omega)) ; E_P(\omega) = |A_P(\omega)| \quad (6.b)$$

The analytic spectrum or, its estimate the analytic periodogram, has its real part always positive. As a consequence its inverse is also definite positive, from these properties it can be concluded that $A_P(\omega)$ and its inverse are both analytic and minimum phase functions. From the minimum phase it is easy to conclude that given $A_P(\omega)$ we can derive $E_P(\omega)$, and, viceversa, given $E_P(\omega)$ obtaining the minimum phase associated we can obtain $A_P(\omega)$ and $P(\omega)$. In summary, doing spectral estimation we handle the same problem that spectral envelope or analytic spectrum estimation.

Furthermore, in clarifying the claimed statistical stability of the envelope, results from its fourth order function character. To be more clear, the inverse Fourier transform $\phi_P(m)$ of the squared envelope $E_P^2(\omega)$ is related with the causal a.c.f. in an additional autocorrelation form.

$$\phi(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} E_P^2(\omega) \cdot \exp(jm\omega) d\omega \quad (7.a)$$

$$\phi(m) = \frac{1}{N} \sum_{q=0}^{N-|m|} r(q) \cdot r(q+m) \quad (7.b)$$

From (7.b) it could concluded the smooth behaviour of the periodogram envelope and its statistical stability when compared with the periodogram.

The formula (7.b) provides the way out to compute the periodogram envelope. Nevertheless, computing the magnitude of the Fourier transform of the causal acf is the fast way to obtain the target function.

Finally, note that in evaluating the periodogram envelope the lag-window used for $r_C(n)$ has no consequences on the positive character of $E_P(\omega)$. This in an additional advantage of using $E_P(\omega)$ as final estimate or first order information in any spectral estimation procedure.

3. SPECTRAL ENVELOPE AND MODELS

In this section we will examine the incidence of signal models on the envelope representation.

Assuming the general case of an ARMA model for the random process under analysis, the acf can be represented by the quotient of two polynomials $C(z)$ and $D(z)$.

$$R(z) = \frac{C(z)C(z^{-1})}{D(z)D(z^{-1})} = \frac{H(z)}{D(z)} + \frac{H(z^{-1})}{D(z^{-1})} \quad (8)$$

When $R(z)$ is decomposed in two quotients which differ in the root location inside and outside the unit circle, it becomes clear that the analytic spectrum $A_S(z)$ can be associated with the first term.

$$R_C(z) = A_S(z) = \frac{H(z)}{D(z)} \quad (9)$$

The other relationships of interest are derived from (8) and (9). The function we have in mind are the power spectrum $S(z)$ and the squared envelope $E(z)E(z^{-1})$.

$$S(z) = R(z) = \frac{C(z)C(z^{-1})}{D(z)D(z^{-1})}; \quad (10.a)$$

$$E(z)E(z^{-1}) = \frac{H(z)H(z^{-1})}{D(z)D(z^{-1})} \quad (10.b)$$

This last formula reveals that the envelope contains the same poles that the actual power spectral density. Also the rational character of $E^2(\omega)$ proves that there is no bandwidth extension from the spectral density to the envelope. This last property suggest that it could be of interest to apply all-pole modelling to the envelope, which will have more sense than over the power spectrum mainly for speech recognition applications. But, let us summarize which are the reasons behind the interest of the envelope representation.

In variational procedures for spectral estimation there are two different functions, the objective to be extremized and the constraints. As claimed in [2] the crucial choice for the resulting quality is the kind of constraints to be handled. The objective function dictates only the way the information contained in the constraints is used or the signal model selected.

Thus, it is clear that, regardless the signal model or the objective, the key choice is the constraints we set in finding an extremum for a given objective. To be more concreat, the way to derive constraints is always from an associated function to the periodogram as it is shown in (11).

$$\rho(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(P(\omega)) \exp(jn\omega) d\omega \quad (11)$$

Note that for $\psi(.)=0$ we have the classical autocorrelation constraints. When dealing with additional functions $\psi(.)$ to add more constraints to the variational procedure it is necessary to select $\psi(.)$ with the following features:

- Do not introduce redundancy with correlation constraints.
- Do not use non linear functions which produce bandwidth extension.

- Use function which produce constraints $\rho(n)$ with the highest statistical stability.
- Do not introduce highly non-linear problems to be faced in searching procedures to solve for the parameters defining the Lagrangian.

It is clear that envelope constraints are a valid candidate for additional constraints for a variational procedure.

Envelope constraints:

$$\phi(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} A_S(\omega) A_S^*(\omega) \exp(jm\omega) d\omega; \quad m=1, Q \quad (12.a)$$

Correlations constraints:

$$r(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} A_S(\omega) \exp(jn\omega) d\omega; \quad n=0, Q \quad (12.b)$$

At this moment, the objective have to be selected in order to obtain a desired signal model. When an ARMA model is the choice, the objective must be:

$$\phi^*(0) = L = \frac{1}{2\pi} \int_{-\pi}^{\pi} A_S(\omega) A_S^*(\omega) d\omega \quad (13)$$

Forming the Lagrangian and setting to zero the derivatives with respect the objective $A_S(\omega)$ a rational model is obtained for the analytic spectrum estimate.

$$A_S(z) = \frac{B(z)}{D(z)} \quad (14)$$

Where both polynomials are minimum phase. The final spectral estimate is derived by taking the real part of $A_S(j\omega)$.

4. ALGORITHM FOR ARMA SPECTRAL ESTIMATION

In facing the problem of finding the coefficients $b(q)$ and $d(q)$ of the polynomials $B(z)$ and $D(z)$ the ideas reported by Mullis and Roberts [3] for mixed first and second order information in filter design will be used.

First at all, note that in order to hold for the correlation constraints, due to the structure of $A_S(z)$, the following set of equations can be set:

$$A_S(z) = \frac{B(z)}{D(z)} = R_C(z) \quad (15)$$

where the first $Q+1$ lags of $R_C(z)$ are the given data autocorrelation sequence including the zero lag. Thus rewriting (15) as $B(z)=R_C(z).D(z)$ and because $r_c(n)$ is a causal sequence we obtain (16).



$$\underline{b} = \underline{R}_C \underline{d} \tag{16}$$

Where

$$\underline{b}^T = [b(0), b(1), \dots, b(Q)] \tag{17.a}$$

$$\underline{d}^T = [d(0), d(1), \dots, d(Q)] \tag{17.b}$$

$$\underline{R}_C = \begin{matrix} r(0)/2 & 0 & 0 & \dots & 0 \\ r(1) & r(0)/2 & 0 & \dots & 0 \\ r(2) & r(1) & r(0)/2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ r(Q) & r(Q-1) & r(Q-2) & \dots & r(0)/2 \end{matrix} \tag{17.c}$$

and symbol τ indicates transpose.

Going to the envelope constraints, we can see that, naming $\Phi(z)$ to the z-transform of the measured $\phi(m)$ and obtained from the periodogram using (18), it is

$$\phi(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} E_P^2(\omega) \exp(jm\omega) d\omega; m=0, Q \tag{18}$$

possible to derive equation (19) for the envelope constraints.

$$A_S(z^{-1}) \frac{B(z)}{D(z)} = \Phi(z) \tag{19}$$

Before going further in the algorithm, it is worthwhile to remark two aspects of the above formulation. First, note that, both $\Phi(z)$ and $R_C(z)$, are z-transforms of the corresponding sequences containing Q+1 samples from the periodogram, and the rest are the extrapolated values by the analytic spectral model. Second, the lag zero of the sequence $\phi(.)$ which forms $\Phi(z)$ is just the value to be extremized (see (13)); to state this difference we will denote the optimum value as $\phi^*(0)$, which has a different value that the $\phi^*(0)$ obtained from (18). We will be back in a moment over this point.

Thus, because $R_C(z^{-1})$ is equal to $A_S(z^{-1})$ (see (15)), we can obtain the following matrix formula for (19).

$$\underline{R}_C^T \underline{b} = \underline{\Phi} \underline{d} \tag{20}$$

Being

$$\underline{\Phi} = \begin{matrix} \phi^*(0) & \phi(1) & \phi(2) & \dots & \phi(0) \\ \phi(1) & \phi^*(0) & \phi(1) & \dots & \phi(Q-1) \\ \phi(2) & \phi(1) & \phi^*(0) & \dots & \phi(Q-2) \\ \dots & \dots & \dots & \dots & \dots \\ \phi(Q) & \phi(Q-1) & \phi(Q-2) & \dots & \phi^*(0) \end{matrix} \tag{21}$$

Finally, by substitution of (16) in (20) we achieve the

resulting equation to be solved in order to find vector \underline{d} and vector \underline{b} .

$$(\underline{R}_C^T \underline{R}_C) \underline{d} = \underline{\Phi} \underline{d} \tag{22}$$

At this moment, it is important to establish that $\underline{\Phi}$ contains an unknown entry (i.e. the objective $\phi^*(0)$). Naming $\underline{\Phi}_0$ the matrix that, with the same structure that $\underline{\Phi}$, contains the measured value $\phi(\omega)$, the difference between them is just shown in (23) being \underline{I} the identity matrix.

$$\phi(0) = \phi^*(0) + \Delta \tag{23.a}$$

$$\underline{\Phi}_0 = \underline{\Phi} + \Delta \underline{I} \tag{23.b}$$

Doing this, equation (23) can be reformulated as (24).

$$(\underline{\Phi}_0 - \underline{R}_C^T \underline{R}_C) \underline{d} = \Delta \underline{d} \tag{24}$$

Based on the fact (by the way they are derived from $A_P(\omega)$) that matrix $\underline{\Phi}_0 - \underline{R}_C^T \underline{R}_C$ must be definite positive, the optimum value for Δ is the minimum eigenvalue of $\underline{\Phi}_0 - \underline{R}_C^T \underline{R}_C$. With this, the coefficients \underline{d} are the coefficient of the minimum eigenvector of (24) and the optimum $\phi(0)$ is just the measured value $\phi(0)$ minus the minimum eigenvalue defined by (24). This completes the algorithm. The algorithm reported here is a short description of the principles contained in reference [3], further details on the algorithm, theorema and properties associated to the mixed second and first order information matrix $\underline{\Phi}_0 - \underline{R}_C^T \underline{R}_C$, can be found in this reference.

6. REFERENCES

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