GENERALIZED WIENER-G FUNCTIONALS FOR NON-LINEAR SYSTEMS APPLICATION TO PARTIALLY COHERENT IMAGING

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RESUME

Le processus de formation d'images 2-D d'objets 3-D en éclairage partiellement cohérent - c'est la cas de la microscopie optique - ne peut être correctement décrit qu'à partir de la transmission de l'intensité mutuelle ou de la densité spectrale mutuelle de l'objet 3-D. Ceci implique un processus non linéaire avec le transfert d'une série de Volterra . Le schéma d'orthogonalisation de Wiener est appliqué à la description de ce processus , mais avec un produit scalaire différent qui est défini comme une moyenne d'ensemble sur une classe d'objets 3-D à imager.Des G-Fonctionnelles Generalisées de Wiener (**GWGFs**) sont introduites et calculées, permettant une représentation orthogonales de la densité spectrale mutuelle de l'objet 3-D. Les noyaux successifs de ces fonctionnelles sont montrés dépendre du noyau leader des séries de Volterra associées. Les **GWGFs** sont ensuite exprimées sur une base de fonctions orthogonales (les fonctions prolates sphéroidales linéaires sont adaptées au cas de la microscopie optique avec des signaux-objets de produit extension x bande passante réduit).On montre que la formation d'images 2-D d'objets 3-D en microscopie est alors correctement décrite dans le cadre de l'algèbre tensorielle.

SUMMARY

We demonstrate that the imaging of a 3-D distribution of index of refraction by a partially coherent optical system involves the transmission of series of Volterra functionals of the object function, that represent either the mutual intensity or the image intensity. According to Wiener, an optimal description of this non-linear process is based on the definition and calculation of series of orthogonal functionals. When applied to partially coherent imagery this approach requires the introduction of a scalar product different from that used by Wiener for temporal signals. A set average that characterizes the spatial - or spectral - properties of a specific class of optical objects to be imaged is proposed. Up to the third moments of the object class are shown to be necessary to account for the transmisssion of 3-D informations in partially coherent imaging. They are supposed to be available to work out the functional orthogonalisation scheme. As a consequence, generalized Wiener G-functionals (**GWGFs**) are derived , that yield an orthogonal representation of the mutual spectral density, and the image spectral density . Similarly to Wiener's work on the class of white, Gaussian temporal signals, the successive kernels of the **GWGFs** have the property of being expressed in terms of the leading kernel of each **GWGF**. But the symetry features of Wiener G-functionals do no longer hold here.

The next and necessary step in the implementation of **GWGFs** is their expression into series of orthogonal functions. At the level of modeling adopted in this first paper it is not useful to specify this basis. Orthogonal series expansions of the **GWGFs** of both the mutual spectral density and the image spectral density are provided. Within this framework the object is described as a vector , and the imaging system is characterized either by signal-independent matrices and tensors (for the transfer of the mutual spectral density) or by signal-independent vectors, matrices and tensors (for the transfer of the image spectral density).

1°) 3-D OBJECTS AND THEIR 2-D IMAGES IN PARTIALLY COHERENT ILLUMINATION : VOLTERRA IMAGING SYSTEMS

HOPKINS has shown more than 30 years ago that the object/image relationship in optical microscopy is based on the transfer of the mutual intensity (or the mutual spectral density) and not on that of the intensity (or the Fourier spectrum). This fact seems to be ignored in so called "quantitative microscopy", but it plays a capital role in the description of the imaging of 3-D objects, as shown by **STREIBL**. The basic equations to be used link the mutual intensies impinging the object, leaving the object, and leaving the imaging system, viz. $\mathbf{J_{in}}(r_1,r_2)$,$\mathbf{J_{obj}}(r_1,r_2)$,and $\mathbf{J_{img}}(r_1,r_2)$. Let us recall them: an optical object being described by a 3-D distribution of the index of refraction,n(r), is also expressed by a normalized function: $v(r) = k^2(1 - n^2(r))$, $k = 2p/\lambda$, λ the wavelength illumination. Assuming that the object is slowly varying compared with l ,a non homogemeous differential equation can be written for the ligth amplitude u(r): $(\nabla^2 + k^2)u(r) = v(r)u(r)$. Then the mutual intensity function, $J(r_1,r_2)$, is shown to follow the double equation: $(\nabla_i^2 + k^2)J(r_1,r_2) = v(r_i)J(r_1,r_2)$, $i = 1, 2$. Introducing the Green function, G(r) , of the homogeneous Helmholtz equation allows to express the mutual intensity that emerges from the object, J_{obj} ,as:

$$J_{obj}(r_1,r_2) = J_{in}(r_1,r_2) + \int dr'_1 G(r_1,r'_1)V(r'_1)J(r'_1,r_2) + \int dr'_2 G^*(r_2,r'_2)V^*(r'_2)J(r_1,r'_2) +$$

$$+ \int \int dr'_1 dr'_2 V(r'_1)V^*(r'_2)G(r_1,r'_1)G^*(r_2,r'_2)J(r'_1,r'_2)$$

where $\mathbf{J_{in}}(r_1,r_2)$ denotes a solution of the homogeneous Helmholtz equation, the mutual intensity that propagates when there is no object (i.e. the illuminating mutual intensity). The mutual intensity which leaves the object is transmitted by an imaging system that yields the mutual intensity , $\mathbf{J_{img}}(r_1,r_2)$,in the image space. The global behavior of the imaging system can be accounted for by the classical relation :

$$J_{img}(r_1,r_2) = \int \int dr_1 dr_2 J_{obj}(r_1,r_2)K(r_1, r_1)K^*(r_2, r_2)$$

where $\mathbf{K}(r,r)$ denotes the coherent spread function of the imaging system.

This expression can be written as a Volterra series if the object function $\mathbf{v}(r)$ is real (i.e. for a non absorbing object):

$$J_{img}(r_1,r_2) = h_0(r_1,r_2) + \int dr' v(r')h_1(r_1, r_2,r') + \int \int dr'_1 dr'_2 V(r'_1)V^*(r'_2)h_2(r_1, r_2,r'_1,r'_2)$$

with the Volterra kernels :

$$h_0(r_1, r_2) = \int dr_1 dr_2 K(r_1,r_1)K^*(r_2,r_2)J_{in}(r_1,r_2), h_1(r_1,r_2,r') = R(r_1-r')T(r_2,r') + R^*(r_2-r')T^*(r_1,r')$$

$$h_2(r_1, r_2,r'_1,r'_2) = R(r_1-r'_1)R^*(r_2-r'_2)J_{in}(r'_1,r'_2) ,$$

$$R(r-r') = \int dr' G(r-r')K(r-r') \text{ , and } \qquad T(r,r') = \int dr\, K^*(r-r)J_{in}(r',r)$$

Assuming the object to be real, its Fourier transform ,$\mathcal{V}(u)$, is hermitian . The image mutual spectral density then appears as a Volterra series: $\mathcal{J}_{img}(u_1,u_2) = \mathcal{H}_0 + \mathcal{H}_1(\mathcal{V}(u)) + \mathcal{H}_2(\mathcal{V}(u))$

i.e. $\mathcal{I}_{img}(u_1,u_2)= \mathfrak{h}_0(u_1,u_2)+ \int du\mathcal{V}(u)\, \mathfrak{h}_1(u_1,u_2,u) + \int\int du_1' du_2' \mathcal{V}(u_1')\mathcal{V}^*(u_2')\, \mathfrak{h}_2(u_1-u_1',u_2-u_2')$

with the Volterra kernels :

$$\mathfrak{h}_0(u_1,u_2) = \mathcal{I}(u_1,u_2)\mathcal{K}(u_1)\mathcal{K}^*(-u_2)$$
$$\mathfrak{h}_1(u_1,u_2,u) = \mathcal{R}(u_1)\mathcal{K}^*(-u_2)\mathcal{I}_{in}(u_1-u,u_2) + \mathcal{R}^*(-u_2)\mathcal{K}(u_1)\, \mathcal{I}_{in}(u_1,u_2-u)$$
$$\mathfrak{h}_2(u_1,u_2,u_1',u_2')= \mathcal{R}(u_1)\, \mathcal{R}^*(-u_2)\mathcal{I}_{in}(u_1-u_1',u_2-u_2')\,.$$

Within the limited range of the present paper we will concentrate on the analysis of the mutual spectral density. A complete treatment of the transfer of the spectrum - a particular case of the present one- can be found in other publications.

2°)GENERALIZED WIENER G-FUNCTIONAL REPRESENTATION OF THE MUTUAL SPECTRAL DENSITY

As pointed by **SCHETZEN** there are two difficulties in using Volterra series to describe the output of a non-linear system. First, the Volterra series representation may converge for only a limited range of the system input amplitude. Second, it is difficult to measure the Volterra kernels,h_n , since the respective contributions of each of the system kernel can hardly be separated from the total system response. These problems were circumvented by **WIENER** by forming an orthogonal set of functionals, G_n , from the set of H_ns Volterra functionals.Wiener functionals are orthogonal when the input signals belong to the class of white, Gaussian, zero-mean time functions : they are called *G-functionals* . A G-functional is a non homogeneous Volterra functional, $G_n(k_n,\mathbf{v}(\mathbf{x}))$, that has the property of being orthogonal to any Volterra functional , $H_m(\mathbf{v}(\mathbf{x}))$, of degree m less than n :

$$< G_n(k_n),v(x))\, H_m(v(x)) > = 0$$

where **< >** denotes the average value with respect to the variable y. The functional is expressed as:

$$G_n(k_n,v(x)) = k_{0(n)} + \sum_{p=1}^{n} \int \dots \int dx_1 \dots dx_p k_{p(n)}(x_1, \dots ,x_p,y)v(x_1) \dots v(x_p)$$

where $k_{p(n)}(\mathbf{x}_1, \dots ,\mathbf{x}_p,y)$ is the p^{th} order Wiener kernel of G_n. Wiener kernels $k_{p(n)}$, p= 0, ... ,n-1, are shown to be determined uniquely from the leading kernel, $k_{n(n)}$. Moreover they exhibit nice rules of parity, due to the Gaussian properties of the considered input signals.The next step in Wiener theory of G-functionals consists of expressing the kernels $k_{p(n)}$ using orthogonal functions.

2°-a) GWGFs of the mutual spectral density

Since optical objects are not white,Gaussian, zero mean signals we introduce in the fulfillment of the orthogonality condition a scalar product that represents a *set average* over a class of objects to be imaged. A straithforward but cumbersome computation yields the **GWGF** representation:

$$\mathcal{I}(u_1,u_2) = \sum_{p=0}^{2} \mathcal{G}_p(\mathfrak{k}_{pp} ,\mathcal{V}(u))$$

* $\underline{\mathcal{G}_0(\mathfrak{k}_{00},\mathcal{V}(u))} = \mathfrak{k}_{00}(u_1,u_2) = \mathfrak{h}_0(u_1,u_2) - \mathfrak{k}_{01}(u_1,u_2) - \mathfrak{k}_{02}(u_1,u_2)$

* $\underline{\mathcal{G}_1(\mathfrak{k}_{11},\mathcal{V}(u))} = \mathfrak{k}_{01}(u_1,u_2) + \int du\mathcal{V}(u)\, \mathfrak{k}_{11}(u_1,u_2,u)$, $\mathfrak{k}_{01}(u_1,u_2) = - \int du\mathcal{M}_1(u)\, \mathfrak{k}_{11}(u_1,u_2,u)$

$$\mathfrak{k}_{11}(u_1,u_2,u) = \mathfrak{h}_1(u_1,u_2,u) - \mathfrak{k}_{12}(u_1,u_2,u)$$

* $\underline{\mathcal{G}_2(\mathfrak{k}_{22},\mathcal{V}(u))} = \mathfrak{k}_{02}(u_1,u_2) + \int du\mathcal{V}(u)\, \mathfrak{k}_{12}(u_1,u_2,u) + \int\int du_1' du_2' \mathcal{V}(u_1')\mathcal{V}^*(u_2')\mathfrak{k}_{22}(u_1,u_2,u_1',u_2')$,

with: $$\mathcal{K}_{02}(u_1,u_2) = -\int\int du_1' du_2' \mathcal{K}_{22}(u_1',u_2',u_1,u_2) \frac{[\mathcal{M}_3(u_1',u_1',u_2')\mathcal{M}(u_1')\mathcal{M}_2^2(u_1',u_2')]}{[\mathcal{M}_2(u_1',u_2') - \mathcal{M}_1^2(u_1')]}$$

$$\mathcal{K}_{12}(u_1,u_2,u) = -\int du' \; \mathcal{K}_{22}(u_1,u_2,u,u') \frac{[\mathcal{M}_3(u,u,u') - \mathcal{M}_2(u,u')\mathcal{M}_1(u)]}{[\mathcal{M}_2(u,u) - \mathcal{M}_1^2(u)]}$$

$$\mathcal{K}_{22}(u_1,u_2,u_1',u_2') = \mathcal{H}_2(u_1-u_1',u_2-u_2')$$

with the object class moments:
$\mathcal{M}_1 = \langle \mathcal{V}(u) \rangle$, $\mathcal{M}_2(u_1',u_2') = \langle \mathcal{V}(u_1') \mathcal{V}^*(u_2') \rangle$, $\mathcal{M}_3(u,u,u') = \langle \mathcal{V}(u) \mathcal{V}(u) \mathcal{V}(u') \rangle$.

2-b) Orthogonal series expansion of the mutual spectral density GWGFs .

Using a complete set of orthogonal functions, $\mathbf{P_n}$, the object spectrum is described by the vector **a** made up of its coefs expansion ; in the same way the **GWGF** kernels can be shown to admit a Schmidt-type expansion:

$$\mathcal{K}_{0i}(u_1,u_2) = \sum_{n_{0i}} \sum_{m_{0i}} a^{(i)}_{n_{0i}m_{0i}} \; \mathcal{P}_{n_{0i}}(u_1)\, \mathcal{P}_{m_{0i}}(u_2),$$ i=0, ... ,2 for the first kernels of $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_2$

$$\mathcal{K}_{1j}(u_1,u_2,u) = \sum_{n_{1j}} \sum_{m_{1j}} \sum_{l_{1j}} b^{(j)}_{n_{1j}m_{1j}l_{1j}} \mathcal{P}_{n_{1j}}(u_1)\mathcal{P}_{m_{1j}}(u_2)\, \mathcal{P}_{l_{1j}}(u),$$ j= 1,2 for the 2nd kernels of $\mathcal{G}_1$ and $\mathcal{G}_2$

$$\mathcal{K}_{22}(u_1',u_2',u_1,u_2) = \sum_{n_{22}} \sum_{m_{22}} \sum_{l_{22}} \sum_{q_{22}} g_{n_{22}m_{22}l_{22}q_{22}} \; \mathcal{P}_{n_{22}}(u_1')\mathcal{P}_{m_{22}}(u_2')\; \mathcal{P}_{l_{22}}(u_1)\mathcal{P}_{q_{22}}(u_2)$$

for the third kernel of $\mathcal{G}_2$. A little algebra then yields the folllowing orthogonal representation of the **GWGFs** of the mutual spectral density: $\mathcal{J}(u_1,u_2) = \sum_i \sum_j d_{ij}P_i(u_1)P_j(u_2)$ with :

the spectral density matrix : $D = (d_{ij})$: $D = D^{(0)} + D^{(1)} + D^{(2)}$,with :

$D^{(0)} = A^{(00)}$, from **GWGF** $\mathcal{G}_0$, $D^{(1)} = A^{(01)} + a^t f B^{(11)}$, from **GWGF** $\mathcal{G}_1$

$D^{(2)} = A^{(02)} + a^t f B^{(12)} + a^t f C^{(22)} f a$, from **GWGF** $\mathcal{G}_2$.

Matrices $A^{(0i)}$,i=0,...,2, 3rd order tensors $B^{(1j)}$, j= 1,2 , and 4th order tensor $C^{(22)}$ are obtained from the orthogonal representations of the kernels of the **GWGFs**. Notice that $D^{(2)}$ shows a generalized bilinear form of the object spectral density vector **a** .

Applications to the description of image formation by a microscope using a linear algebra and tensor scheme based on the use of Prolate Spheroidal Wave Functions is under publication elsewhere.

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