ral/frequency version of SST can be performed as a preprocessing to a temporal also version of HRT, yielding a matrix with rank equal to the number of replicas in the observations. This processing is dual of the conventional SST, and so, it requires a condition that is dual of the condition given above: $N \geq 2P$, where now N is the Time/Bandwith product of the received signal, for the case of P coherent replicas of a single source.

Methods that consider joint time/space processing of the received waveform are also available. One such technique is CSSM [7], which resorts to frequency smoothing to eliminate the ambiguity due to the possible coherency of the sources. This method is derived in a very general setting, where only asymptotically (when the number of frequencies tends to infinity) is the ambiguity eliminated. However, if we restrict attention to the case of equispaced processing frequencies, and to multipath propagation of several radiating sources, bounds on the number of detectable sources can be found, which are a function of both the array size and the number of temporal degrees of freedom in the waveform.

The paper is organized as follows: In section 2, we state two Facts concerning the properties of smoothed matrices that will be repeatedly used to establish the bounds mentioned before. In section 3, we derive bounds for SST, and give a geometric interpretation of the general bound in terms of the dimensionality of the intersection of possible estimated signal subspaces. Section 4 is devoted to bounds for the CSSM.

A detailed version of the work reported here will be presented elsewhere.

2 Basic Facts

Let A be the matrix that represents in a given basis the linear transformation $\mathcal{A}$ from $\mathbf{C}^q$ into itself, where $\mathbf{C}$ is the field of complex numbers. A subspace $\mathcal{U} \subseteq \mathbf{C}^q$ is said to be an invariant subspace of A iff

$$A\mathcal{U} \subset \mathcal{U}, \quad \Longleftrightarrow \quad Av \in \mathcal{U}, \quad \forall v \in \mathcal{U}. \tag{1}$$

It can be proved (see, e.g. [1]), that invariant subspaces of diagonal matrices of distinct diagonal entries are those generated by sets of euclidean vectors (with a single component different from zero), i.e., an element of a proper invariant subspace of a diagonal matrix must have at least one of its components equal to zero.

From this, and the fact that the eigenvectors of a semidefinite positive matrix which are not in its null space cannot all belong to the same invariant proper subspace of a diagonal matrix in the conditions above, the following can be proved:

Fact 1 ([5]) *Let $S \geq 0$ be a $(P \times P)$ Hermitean matrix of rank r with nonzero diagonal entries, and D a $(P \times P)$ diagonal matrix with distinct nonzero diagonal entries: $D_{ii} \neq D_{jj}, i \neq j$. Then, for $M > P - r$ the matrix $\overline{S}$ defined by,*

$$\overline{S} = \sum_{k=1}^{M} D^{k-1}\, S\, (D^H)^{k-1} \tag{2}$$

has full rank P.

□

Besides its semi-definite positiveness, this bound assumes no particular structure for S. When S is block diagonalizable by a permutation matrix, a tighter bound can be found:

Fact 2 ([5]) *Let D be a diagonal matrix with distinct nonzero diagonal entries, and S a $(P \times P)$ Hermitean matrix with nonzero diagonal entries which is block-diagonalizable by a permutation matrix. Further, let ℓ be the number of diagonal blocks, n_i be the dimension of the i-th block, and r_i its rank. For*

$$M > \max_{i=1,\ldots,\ell} (n_i - r_i), \tag{3}$$

the averaged matrix given by Eq.(2) has full rank P.

□

In the following sections, these two facts are used to derive the bounds for the number of sources detectable by specific smoothing techniques.

3 Narrowband/Spatial Processing

Conventional narrowband (NB) HRT are based on an orthogonal decomposition of the observation space in signal and noise subspaces. This decomposition is obtained from eigendecompostion of the sample covariance matrix that, asymptotically, has the following form:

$$R = ASA^H + \sigma^2 \Sigma \tag{4}$$

where A is the steering $(K \times P)$ matrix, S is the $(P \times P)$ source vector covariance matrix, and Σ is the $(K \times K)$ noise contribution covariance matrix.

SST is based on averaging the sample covariance matrix over contiguous subarrays, yielding a "smoothed" source covariance matrix [6] that can be described by Eq.(2), where now, M is the number of subarrays,

$$D = \mathrm{diag}\{e^{j\omega_0\tau_1} \cdots e^{j\omega_0\tau_P}\}, \tag{5}$$

P is the number of impinging replicas, and S is the $(P \times P)$ covariance matrix of the source vector:

$$S = E[s(t)s^H(t)]. \tag{6}$$

The goal of SST is to replace S, which for perfectly correlated sources is a singular matrix, with $\overline{S}$, which, for conveniently large M will have rank equal to the dimension of S, independently of its rank. Using Fact 1, we conclude that for $M > P - r$, $\overline{S}$ has full rank P. To obtain the bound on the number of sensors, we use the fact that with an array of K sensors, and a subarray size of q, the number of subarrays is $K - q + 1$. Since each subarray must meet the resolvability condition of the HR methods (once it is the size of the "virtual array" seen by the HR method), the following bound is obtained:

$$K > 2P - r \tag{7}$$

This is the general condition that must be met to ensure that SST yields a full rank source covariance matrix.

Geometric Framework

Consider a uniform linear array of K sensors. Let Ω be the set of possible DOA's. The array manifold is defined as the linear subspace:

$$\mathcal{A} = \mathrm{Sp}\{a(\theta) | \theta \in \Omega\}. \tag{8}$$

It is said that the array manifold has no ambiguities if the application from Ω to $\mathcal{A}$ is one-to-one. This guarantees that the DOA can be uniquely inferred from knowledge of the steering vector. We study below another kind of ambiguity, related to the inference of a set of P DOA's from the observed signal subspace (defined in the sense of the HRT's). Namely, we say that there is no ambiguity in the estimation of P DOA's when the sets of possible estimates of the signal subspace corresponding to distinct sets of P DOA's are distinct.

Let Ω_P be the subset of the P-dimensional cartesian product Ω^P, defined by:

$$\Omega_P = \{\Theta = [\theta_1 \theta_2 \ldots \theta_P] \in \Omega^P \mid \theta_i \neq \theta_j, i \neq j\} \tag{9}$$

and let Υ^P be the quotient set of Ω_P on the following equivalence relation:

$$\Theta^1 \equiv \Theta^2, \quad \Theta^1, \Theta^2 \in \Omega_P \Longleftrightarrow \forall i, \exists j, \Theta^1_i = \Theta^2_j \tag{10}$$

The set Υ^P is the set of all possible P-tuples of directions of arrival for P distinct DOA's, irrespective of ordering.

Let

$$\mathcal{A}(\Theta) = \mathrm{Sp}\{a(\theta_1), \ldots, a(\theta_P)\}. \tag{11}$$

and $\mathcal{A}_P = \{\mathcal{A}(\Theta), \Theta \in \Upsilon^P\}$.

Denote by $\hat{\mathcal{S}}$ the observed signal subspace, for a given set Θ^a. It is known that $\hat{\mathcal{S}} \subseteq \mathcal{A}(\Theta^a)$. Denote by $\mathcal{A}^q(\Theta)$ the linear span of the Vandermonde basis of $\mathcal{A}(\Theta)$ excluding $a(\theta_q)$:

$$\mathcal{A}^q(\Theta) = \mathrm{Sp}\{a(\theta_1), \ldots, a(\theta_{q-1}), a(\theta_{q+1}), \ldots a(\theta_P)\} \tag{12}$$

It can be shown that $\hat{\mathcal{S}} \not\subseteq \mathcal{A}^q(\Theta^a), q \in \{1, \ldots, P\}$.

Let $A(\Theta)$ denote the matrix whose columns are the steering vectors corresponding to the DOA's in Θ. Define [1]

$$\delta_P(\Theta^a, \Theta) \triangleq \dim \mathrm{Ker}\, [A(\Theta^a \cup \Theta)]\,. \tag{13}$$

If $\forall \Theta \neq \Theta^a, \delta_P(\Theta^a, \Theta) = 0$, no observed signal subspace can be a subset of both $\mathcal{A}(\Theta^a)$ and $\mathcal{A}(\Theta)$, i.e., $\hat{\mathcal{S}}$ uniquely determines $\mathcal{A}(\Theta^a)$. There is no ambiguity in the estimation of Θ^a and Θ. If $\delta_P(\Theta^a, \Theta) > 0$ there are nontrivial subspaces of $\mathcal{A}(\Theta^a) \cap \mathcal{A}(\Theta)$, besides those that are generated by a subset of the Vandermode basis of both spaces, expressing the existence of ambiguities in the estimation of Θ^a.

Let n be the dimension of $\Theta^a \cup \Theta$. From the properties of Vandermonde vectors,

$$\delta_P(\Theta^a, \Theta) = \begin{cases} 0, & n \leq K \\ n - K, & n > K \end{cases} \tag{14}$$

[1] we abusively use $\Theta^1 \cup \Theta^2$ to denote the vector formed by all the distinct DOA's present in the two vectors and $\Theta^1 \cap \Theta^2$ the vector formed by the intersection of their entries .

We conclude that for an array of K sensors, the sets of possible signal subspaces corresponding to source configurations with at least $2P - K$ common DOA's have empty intersection. The sets of DOA's for which there may be ambiguity are those that have less than $2P - K$ common DOA's. The case that leads to an ambiguity subspace of larger dimension is $n = 2P($ i.e. $\Theta^a \cap \Theta = \{\ \})$. For this case,

$$\dim(\mathcal{A}(\Theta^a) \cap \mathcal{A}(\Theta)) = 2P - K. \tag{15}$$

If the observed signal subspace is required to have dimension greater than this value, then it can only be a subset of the corresponding $\mathcal{A}(\Theta^a)$, and unicity is maintained.

Let now $\mathcal{F}$ be a set of possible covariance matrices, and denote by $\mathcal{A}_{\mathcal{F}}(\Theta)$ the set of all possible estimates of the signal subspace for sources with covariance matrices $S \in \mathcal{F}$ for a given Θ. Define $\mathcal{A}_{\mathcal{F}} = \{\mathcal{A}_{\mathcal{F}}(\Theta) | \Theta \in \Upsilon^P\}$.

We say that there is no ambiguity in the estimation of P sources, with covariance matrix in the set $\mathcal{F}$, if

$$\mathcal{A}_{\mathcal{F}}(\Theta^a) \cap \mathcal{A}_{\mathcal{F}}(\Theta) = \{\ \}, \quad \Theta \neq \Theta^a. \tag{16}$$

Consider the set $\mathcal{F}^* = \{S | \rho(S) > 2P - K\}$. Since the dimension of the signal subspace equals the rank of the covariance matrix, the elements of $\mathcal{A}_{\mathcal{F}}$ corresponding to distinct Θ's are disjoint, and there is there is a one-to-one relation between the family $\mathcal{A}_{\mathcal{F}^*}$ and the set of Υ^P. Note that the condition obtained here, $\rho(S) > 2P - K$, is the same as (7). However, we stress that while this bound coincides with the resolvability condition of SST, this derivation is independent of any processing technique, relying solely on the geometric properties of the array.

Groups of Independent Sources

When there are uncorrelated sources, each one propagating to the receiver through multiple paths, the source covariance matrix is in the conditions of Fact 2, and the following bound results:

$$K > P + \max_i (n_i - r_i) \tag{17}$$

where n_i is the number of replicas received from source i and r_i is the rank of their correlation matrix.

4 Spatial/Temporal Processing: CSSM

In this section we derive a resolvability condition for CSSM [7]. This method assumes no particular array configuration. It is based on the determination of the signal subspace at each frequency component of the received signal, and posterior mapping of each such subspace into the signal subspace at a reference frequency, assuming that initial estimates of the directions of arrival of the replicas are available. The mapped subspaces are then averaged, yielding a "smoothed" covariance matrix, which must be a full rank matrix. Until now, the nonsingularity of the "smoothed" covariance matrix has been assured only asymptotically (as the number of frequency components tends to infinity). To derive the resolvability condition given here, we assume the following:

i) frequency structure: The frequency analysis is carried out at equally spaced frequencies:

$$\omega_n = \omega_0 + n\Delta\omega \tag{18}$$

ii) covariance structure: The vector of source signals has the following structure:

$$\begin{array}{l} s(t)^T = [\ s^1(t)^T \dots \ s^\ell(t)^T\ ] \\ \left[s^i(t)^T\right]_j = a_{ij} s_i(t - \tau_{ij}) \end{array} \tag{19}$$

where the signals $s_i(t)$ are uncorrelated, and τ_{ij} is the propagation delay of source i to the reference sensor of the array, through path j.

Condition (18) mimics in the frequency domain the uniform linear array assumption of SST, whereas condition (19) sets us in the conditions of Fact 2.

We model the delays $\{\tau_{ij}\}$ as deterministic quantities, and the attenuation factors $\{a_{ij}\}$ as random variables, which allows us to introduce some partial decorrelation between replicas of the same signal received over distinct paths.

The smoothed source covariance matrix of CSSM is [7]:

$$\overline{S} = \sum_{n=1}^{N} S_n \tag{20}$$

where S_n is the source covariance matrix at frequency ω_n,

$$S_n = E[S(\omega_n)S(\omega_n)^H] \tag{21}$$

and N is the number of frequency components (the time/bandwidth product).

Under hypothesis **ii)**, S_n is a block diagonal matrix:

$$S_n = \begin{bmatrix} W_1 & & \underline{0} \\ & \ddots & \\ \underline{0} & & W_\ell \end{bmatrix} \tag{22}$$

Each block W_i in (22) is an $(n_i \times n_i)$ matrix.

Assuming a large time/bandwidth product, the Fourier Transform of the rearranged signal at frequency ω_n is approximately given by:

$$S(\omega_n) = \begin{bmatrix} S^1(\omega_n) \\ \vdots \\ S^\ell(\omega_n) \end{bmatrix} \tag{23}$$

where

$$S^i(\omega_n) = S_i(\omega_n) \begin{bmatrix} e^{j(\Delta\omega)\tau_{i1}} & & \underline{0} \\ & \ddots & \\ \underline{0} & & e^{j(\Delta\omega)\tau_{in_i}} \end{bmatrix}^n \begin{bmatrix} a_{i1}e^{j\omega_0\tau_{i1}} \\ \vdots \\ a_{i\ell}e^{j\omega_0\tau_{in_i}} \end{bmatrix} \tag{24}$$

Denote by D^i the $(n_i \times n_i)$ diagonal matrix in this equation, and by b_i the vector it multiplies. Then,

$$W_i = \sigma_i^2(\omega_n)(D^i)^n B_i (D^i)^{n^H} \tag{25}$$

where $B_i = E[b_i b_i^H]$ has rank r_i. Using the expression for W_i in (20) we get, for the i-th diagonal block of $\overline{S}$:

$$[\overline{S}]_i = \sigma_i^2(\omega_n) \sum_{n=1}^{N} (D^i)^n B_i (D^i)^{n^H} \tag{26}$$

which has the same structure as the covariance matrix of SST. The following fact then holds:

Fact 3 *Under hypotheses* **i)** *and* **ii)** *above, and if :*

- *the random variables a_{ij} have non-zero mean-square value (non zero power condition)*
- *the delays τ_{ij} are such that $e^{j\omega_0(\tau_{in}-\tau_{im})} \neq 1, n \neq m$, (distinct diagonal entries of D^i)*

Then, for

$$N > \max_{i=1,\dots,\ell}(n_i - r_i) \tag{27}$$

the matrix $\overline{S}$ has full rank $P = \sum_i n_i$. □

This fact implies the following resolvability condition for this method:

$$\left\{ \begin{array}{l} K > P \\ N \geq max(n_i - r_i) \end{array} \right. \tag{28}$$

For a single propagating source, the following condition is obtained:

$$\left\{ \begin{array}{l} K > P \\ N \geq P - r \end{array} \right. \tag{29}$$

where $r = \rho(B)$.

As it could be expected, given the separate treatment of the spatial and temporal domains, this algorithm has hard, separate conditions on the number of degrees of freedom required in each one.

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