# THE SHAPE INFORMATION DISTRIBUTION ON A THREE-DIMENSIONAL OBJECT

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## RESUME

Les courbes et les surfaces a trois dimension peuvent etre characterisé par leur courbatures principale. Un nombre de chercheurs ont utilisé cette parametrisation numerique de l'image pour l'identification des formes. Pour une fonction de surface continue, les courbatures principale peuvent étre derivé par des calculs locaux; Il est aussi possible de reconstruire exactement la surface si on avait les courbatures principale en chaque point de la surface. Pour une image numerique avec enchantillonment spatiale limité et quantitisation en profondeur, une courbe ou une surface a trois dimension ne peuvent pas etre representer exactement. Dans ce manuscrit un theoreme d'echantillonment pour les surface a pente limité a été developé. Ce theorem et une extension des theoremes de geometrie differentielle pour le cas discret.

Deux nouvelles parametrisations des courbes et des surfaces ont été introduit: la *densité de la forme* identifie les points d'interet et la *distribution de la forme* characterise la courbe ou la surface totale. La densite de la forme peut etre utiliser comme un detecteur de gradients tri-dimensionels pour selectioner la forme essentielle.

## SUMMARY

Three dimensional curves and surfaces may be characterized by their principle curvatures. A number of researchers have used this parameterization of image data for shape identification. For a continuous surface function, the principal curvatures may be derived from local computations; furthermore, it is possible to exactly reconstruct a surface given the principal curvatures at all points on the surface. For a digital depth image with limited spatial sampling and depth quantization, a three dimensional curve or surface cannot be exactly represented. In this paper, a sampling theorem for slope-limited surfaces is developed that extends differential geometry theorems to the discrete case.

Two new parameterizations of curves and surfaces are introduced: the *shape density* identifies points of high shape interest and the *shape distribution* characterizes a total curve or surface. The shape density may be used like a three dimensional gradient (edge) detector to select salient shape identification information from a range image.

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## 1. Introduction

There has been considerable recent interest in the use of differential geometry for object identification[1-9]. This is due to the need for more sophisticated and precise techniques to describe shapes for three dimensional object identification. Differential geometry offers an effective tool for describing the shape of an object.

In this paper, a sampling theorem for a slope-limited surface is derived which extends differential geometry concepts to the discrete case. The calculation of curvature on a continuous surface is reviewed in section 2, and an outline of some basic concepts of a discrete surface is presented in section 3. The constraints, that the sampling must satisfy to represent the shape with respect to the quantization error, are discussed in section 4. The sampling theorem for shape representation is presented, establishing the criteria for using differential geometry for the discrete case. In section 5, the concepts of the *shape density* and the *shape distribution function* are derived, based on differential geometry and information theory. These are used to determine the distribution of the shape information on an object. Finally in section 6, a strategy for 3-D object identification which involves the shape distribution functions is described.

## 2. Curvature on a Continuous Surface

A *surface* in $E^3$ can be written in a parametric form:

$$z = Z(x,y).$$

where $Z(x, y)$ is a differentiable function defined in a region $D$ of the $x, y$ plane.

The *curvature* of a curve at a point $P$ is the limit of the ratio $\Delta\phi/\Delta s$ as another point $Q$ on the curve approaches $P$, where $\Delta\phi$ is the angle between the tangents drawn to the curve at $P$ and $Q$, and $\Delta s$ the arc length of the segment $PQ$ of the curve [10].

The *normal curvature* of a surface in a given direction, is equal to the curvature of the curve, which is obtained by intersecting the surface with a plane perpendicular to the tangent plane and having the given direction. The direction on a surface is called the *principal direction* if the normal curvature of the surface in a direction attains an extremal value. The maximum value $k_1$ and minimum value $k_2$ are called *maximum curvature* and *minimum curvature* respectively. The normal curvature in an arbitrary direction $\theta$ can be specified by:

$$k_\theta = k_1\cos^2\theta + k_2\sin^2\theta.$$

where $\theta$ is the angle with respect to the direction of maximum curvature.

Half the sum of the principal curvatures of a surface

$$H = \frac{1}{2}(k_1+k_2)$$

is called the *mean curvature* of the surface.

The product of the principal curvatures of a surface is called the *Gaussian curvature* of the surface,

$$K = k_1 k_2.$$

It can be shown that, if the surface is defined by the equation $z=Z(x, y)$,

$$H = \frac{1}{2}\frac{(1+Z_y^2)Z_{xx} - 2Z_xZ_yZ_{xy} + (1+Z_x^2)Z_{yy}}{(1+Z_x^2+Z_y^2)^{3/2}}$$

$$K = \frac{Z_{xx}Z_{yy} - Z_{xy}^2}{(1+Z_x^2+Z_y^2)^2},$$

where $Z_x, Z_y, Z_{xx}, Z_{xy}, Z_{yy}$ is the notation for the partial derivatives of the function $Z(x,y)$.

Given the values of $K$ and $H$, The principle curvatures can be obtained from

$$k_{1,2} = H \pm \sqrt{H^2-K}.$$

## 3. Discrete Surfaces

Suppose that a 3-D object is represented by digitized range data, and the data are distributed on a lattice which is periodically arranged in $E^3$. The intervals of the Cartesian coordinates are $\Delta x$, $\Delta y$, and $\Delta z$. Let $d_0$ be the *discrimination distance*, defined as:

$$d_0^2 = (1-\delta(x-x_0))\Delta x^2+(1-\delta(y-y_0))\Delta y^2+(1-\delta(z-z_0))\Delta z^2.$$

where $(x_0,y_0,z_0)$ is a given point and $\delta()$ is the delta function.

**Definition.** The points $(x_i,y_i,z_i)$ and $(x_0,y_0,z_0)$ in $E^3$ are *neighbors* provided that

$$(x_i-x_0)^2+(y_i-y_0)^2+(z_i-z_0)^2 = d_0^2.$$

The set

$$[(x,y,z)\colon (x_i-x_0)^2 + (y_i-y_0)^2 + (z_i-z_0)^2 = d_0^2]$$

is called the *neighborhood* of $(x_0,y_0,z_0)$.

For the 2-D case, we similarly define $d_0$, the discrimination distance, by:

$$d_0^2 = (1-\delta(x-x_0))\Delta x^2+(1-\delta(y-y_0))\Delta y^2.$$

**Definition.** The points $(x_i,y_i)$ and $(x_0,y_0)$ are *neighbors* provided that

$$(x_i-x_0)^2+(y_i-y_0)^2 = d_0^2.$$

The set

$$[(x,y)\colon(x_i-x_0)^2 + (y_i-y_0)^2 = d_0^2]$$

is called the *neighborhood* of $(x_0,y_0)$.

A discrete function, $Z(x_i,y_i)$, with discrete variables is *linked* at the point $(x_0,y_0)$, such that if $(x_i,y_i)$ is any one of the neighbors of the point $(x_0,y_0)$, then

$$|Z(x_i,y_i)-Z(x_0,y_0)| \le N.$$

where $N$ is a given positive number. This link is denoted $L[N]$.

The *ratio of difference* of the function $f(x)$ is defined by:

$$f_x = \Delta f/\Delta x = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$

**Definition.** The ratio of the partial difference of Z with respect to x at $(x_0,y_0)$, denoted by $\Delta Z/\Delta x|_{(x_0,y_0)}$, is the ratio of the difference at $x_0$ of the function of one variable $Z(x,y_0)$. Similarly, the ratio of the partial difference of Z with respect to y at $(x_0,y_0)$, denoted by $\Delta Z/\Delta y|_{(x_0,y_0)}$, is the ratio of the difference at $y_0$ of the function of one variable $Z(x_0,y)$. When Z has a ratio of partial differences at all points of a neighborhood of $(x_0,y_0)$, we can consider the second ratio of partial difference at $(x_0,y_0)$ and so forth. Z is *smooth* at $(x_0,y_0)$ if it has link ratio of partial differences of the second order at $(x_0,y_0)$. Z is a *smooth surface* if it is smooth at all surface points.

An *object O* is considered as the set of points, which we call the inhabitants of the object. The *discrete surface S* is a subset of $O$ in $E^3$, such that for each point of $S$, there at least exists a neighbor which does not belong to the set of $O$. The subset of the object which does not belong to the surface $S$ is defined as the *body*. The *shape* of the object can be considered as the state of its surface in $E^3$.

Image range data records the distance of points from the viewer to the surface of the object, and gives a surface description which is useful for 3-D object identification. From range image data we have no information about the contents of the body of the object. The information of the visible shape is distributed on the surface of the object.

## 4. Sampling a Slope-Limited Surface

A sampling lattice is a periodic arrangement of points in the xy-plane, where the points are defined by the position vectors. Usually, there are an infinite number of curves or surfaces that can be generated by a given set of samples. However, if the slope of the surface is limited and if the samples are taken sufficiently close together in relation to the maximum slope of the surface, then the samples uniquely specify the surface and we can reconstruct it precisely. Furthermore, the theory of differential geometry can be extended from the continuous case to the discrete case, called *discrete differential geometry (DDG)* or *difference geometry*. The following theorem, which establishes the conditions for a slope-limited surface to be represented by its samples, is the base on which the differential geometry is extended to the discrete case.

For convenience of presentation, we will denote $Z(x,y)$ by $Z(\vec{r})$, where $\vec{r}$ is a position vector with coordinates $(x,y)$. Suppose that the surface $Z(\vec{r})$ is differentiable, and the derivative $Z'(\vec{r})$ at $\vec{r}_0$ exists for every direction $\vec{\Delta r}$:

$$Z'(\vec{r_0}) = \lim_{|\vec{\Delta r}|\to 0} \frac{Z(\vec{r_0}+\vec{\Delta r}) - Z(\vec{r_0})}{|\vec{\Delta r}|}.$$

**Definition.** The *slope of a surface* with respect to the point $\vec{r_0}$, is the maximum absolute value of the derivative $Z'(\vec{r_0})$, denoted by $\alpha$,

$$\alpha = \max(abs(Z'(\vec{r_0}))).$$

**Theorem.** Assume that the surface $Z(\vec{r})$ is a slope-limited surface, in which the slope:

$$\alpha \le \alpha_{max}$$

for all except bounded regions. The allowable error is $e$. Provided that

$$\Delta x \le \frac{e}{2\sqrt{2}\alpha_{max}}, \quad \Delta y \le \frac{e}{2\sqrt{2}\alpha_{max}},$$

where $\Delta x, \Delta y$ are the sampling intervals, then $Z(\vec{r})$ is uniquely determined by its samples $Z(m\Delta x, n\Delta y)$ within the allowable error, where $m, n = 0, \pm 1, \pm 2, \cdots$.

**Proof.** Suppose that $\vec{r_0}$ is a point with coordinates $(x_0, y_0)$. $x_0 = n\Delta x$, $y_0 = m\Delta y$. While $\vec{r}$ is a point with the coordinate $(x, y)$. Provided that:

$$n\Delta x \le x \le (n+1)\Delta x, \quad m\Delta y \le y \le (m+1)\Delta y$$

then

$$Z(\vec{r_0}) - \alpha_{max}\Delta r \le Z(\vec{r}) \le Z(\vec{r_0}) + \alpha_{max}\Delta r.$$

where $\Delta r^2 = \Delta x^2 + \Delta y^2$. Therefore:

$$|Z(\vec{r}) - Z(\vec{r_0})| \le \alpha_{max}\Delta r.$$

$$\alpha_{max}\Delta r = \alpha_{max}\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{(\alpha_{max}\,\Delta x)^2 + (\alpha_{max}\,\Delta y)^2}$$

But recalling that:

$$\Delta x \le \frac{e}{2\sqrt{2}\alpha_{max}}, \quad \Delta y \le \frac{e}{2\sqrt{2}\alpha_{max}},$$

yields the final result:

$$|Z(\vec{r}) - Z(\vec{r_0})| \le \sqrt{\frac{e^2}{8} + \frac{e^2}{8}} = \frac{e}{2}.$$

This theorem is suitable for the slope-limited surface with $\alpha < \alpha_{max}$. If $\alpha > \alpha_{max}$, then an exact reconstruction within an error $e$ from samples will not be possible.

If the allowable error is normalized, $e = 1$. Then the sampling intervals should satisfy:

$$\Delta x \le 1/2\sqrt{2}\alpha_{max} \quad \Delta y \le 1/2\sqrt{2}\alpha_{max}.$$

**Corollary.** If the sampling intervals are

$$\Delta x \le \frac{e}{2\sqrt{2}\alpha_{max}}, \quad \Delta y \le \frac{e}{2\sqrt{2}\alpha_{max}},$$

then, the maximum error of the reconstructed surface is $\alpha_{max}\Delta r$.

We can use the concepts and theorems of differential geometry on a discrete surface which satisfies the sampling theorem. If $Z$ is smooth on a discrete surface, then we can use equations derived for the continuous case to calculate the mean, Gaussian, maximum and minimum curvatures.

## 5. Shape Information Distribution

### 5.1. Shape Equivalence

**Definition.** The shapes of two objects in $E^3$ are *identical* provided that their surfaces are congruent. The shapes of two objects in $E^2$ are *identical* provided that their edges are congruent. Therefore, from a mathematical point of view, shape identification is a test of the congruency of surfaces for 3-D objects, or congruency of edges for 2-D objects,

Two surfaces $\Phi$ and $\bar{\Phi}$ are congruent provided there is an isometry $F$ of $E^3$ that carries $\Phi$ exactly onto $\bar{\Phi}$. Thus congruent surfaces have the same shape -- only their positions can be different.

### 5.2. Shape Density

Shape information is not homogeneously distributed on the surface of an object. The human visual system is particularly sensitive to certain regions of an object surface or edge for shape identification purpose. Examples of such regions are corners, holes and boundaries. The sharper bending regions of a surface contribute much more shape information than flat regions do.

We introduce the concept *shape density*, $Ds$, to measure the shape information at a point on the edge or surface. It is reasonable to assume that the shape density $Ds$ should be a function of the principal curvatures:

$$Ds = f(k_1, k_2).$$

where $k_1$, $k_2$ are the maximum and minimum curvatures. Furthermore, it is reasonable to consider that the shape density should increase in value if either of the magnitudes of the orthogonal principal curvatures are increased. Based on these considerations we introduce the following definition for shape density:

**Definition.** If $P$ is a point on a surface, let

$$Ds = \sqrt{(k_1)^2 + (k_2)^2}.$$

where $k_1$ is the maximum curvature and $k_2$ is the minimum curvature of the point $P$ on the surface.

If the minimum curvature $k_2$ is much less the maximum curvature, $k_2 \ll k_1$, then

$$Ds = |k_1|,$$

which means that the shape density is mostly determined by the maximum curvature.

**Definition.** If $P$ is a point on a curve, let

$$Ds = \sqrt{(k_1)^2 + (k_2)^2}.$$

where $k_1$ is the curvature and $k_2$ is the torsion of the point $P$ on the curve. $Ds$ is called the *shape density* of the point on the curve. For the planar curve in $E^2$, $k_2 = 0$; therefore, the shape density is specified by

$$Ds = |k_1|.$$

The shape density at a point is only related to the extent of bending which is the absolute value of the curvature.

For a surface

$$k_1^2 + k_2^2 = (H + \sqrt{H^2 - K})^2 + (H - \sqrt{H^2 - K})^2.$$

therefore:

$$Ds = \sqrt{4H^2 - 2K},$$

where $K$ is the Gaussian curvature and $H$ is the mean curvature.

### 5.3. The Shape Distribution Function

We can use the *shape distribution function* to measure shape information for a given area of a surface or a given length of an edge.

**Definition.** The *shape distribution function* of a given curve is

$$F(r) = \int_{r_1}^{r_2} Ds(r)dr,$$

where $r_1$ and $r_2$ are the start and end points of the curve. For the discrete case,

$$F(r) = \sum_{r_i \in r} Ds(r_i).$$

**Definition.** The *shape distribution function* of a total surface with area $A$ is given by

$$F(A) = \iint_{s \in A} Ds(s)ds.$$

For the discrete case,

$$F(A) = \sum_{s_i \in A} Ds(s_i),$$

where $s_i$ is denoted the point on the surface.

**Definition.** The *shape distribution function* of a subsurface is defined as:

$$F(R) = \iint_{s \in R} Ds(s)ds,$$

where $R$ is denoted the area of the subsurface. For the discrete case,

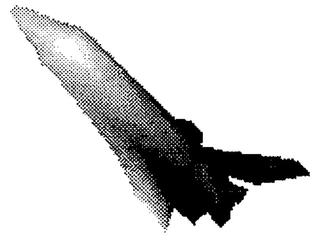

Fig. 1. A synthetic range image of a space shuttle.

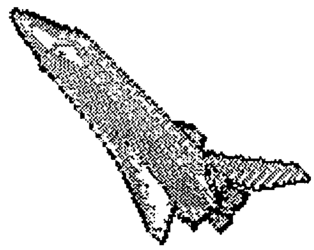

Fig. 2. The shape densities of the space shuttle

Fig. 3. The high shape density regions selected by thresholding. The relative shape distribution function Fr(R) = 0.452.

Fig. 4. The range image of high density regions.

$$F(R) = \sum_{s_i \varepsilon R} Ds(s_i),$$

where $s_i$ is denoted the point belongs to the sub-surface $R$.

The *relative shape distribution function (RSDF)* is defined by

$$F_r(R) = F(R)/F(A).$$

The shape density indicates the amount of shape information at a point on the curve or surface. The shape distribution function measures the amount of shape information associated with an edge or a surface.

## 6. A Strategy for 3-D Shape Identification

We are exploring the following strategy for 3-D shape identification which involves the shape density and shape distribution functions.

1. Pre-process the input range data (filtering and scaling).
2. Calculate the shape density of the input data.
3. Select image elements with high shape density for object identification, i.e., all elements with shape density having a value geater than a threshold $t$. Calculate the shape distribution function (SDF) and the RSDF for all selected image elements. Adjust the threshold $t$ if necessary to obtain a RSDF within a predefined range.
4. Identify the shapes with an evidence accumulating identification scheme such as the Hough Transform.

For example, a range image of a space shuttle is shown in Figure 1. The shape densities of the object are given in Figure 2. The thresholded shape density is shown in Figure 3, in which the threshold of $Ds$ is 9.6, $F(R) = 4173$, $F(A) = 5779$, $F_r(R) = 0.452$. The elements of the range image which have been selected by high shape density values are shown in Figure 4.

The selection of only high shape density elements is important for reducing the computation required for 3-D matching techniques, such as the 3-D Hough transform, to a reasonable amount. In order to increase the confidence of identification, it is necessary to raise the value of the RSDF by decreasing the shape density threshold. For a given threshold, the higher shape density regions are selected and the lower shape density regions are rejected to minimize the computational requirements.

## Conclusion

The shape density function, which is based on the principal curvatures of a surface, has been introduced and its application as an interest operator for shape identification has been outlined. The sampling constraints on a slope-limited surface for accurate representation have been derived. This work provides the foundation for an image data reduction scheme for use with a Hough transform based 3-D object identification system.

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