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Some practical aspects of the affine time-frequency distributions

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RÉSUMÉ

Les représentations temps-fréquence affines sont des outils de description temps-fréquence des signaux à large bande. La référence au groupe affine rappelle leur origine axiomatique fondée sur l'étude des changements d'horloge. L'objet de ce papier est à la fois de présenter une dérivation directe de la principale distribution affine et de donner des indications sur son usage.

ABSTRACT

Affine time-frequency distributions are tools for the tim frequency interpretation of wide-band signals. The referen to the affine group recalls their axiomatic construction whi is essentially based on the study of clock changes. The obje of this paper is both to present a very direct derivation of t main distribution and to give indications for its use.

1 Introduction.

By now, the relevance of the introduction of the affine group of time translations b and dilations $a > 0$ in signal analysis is well established. The action of this group on a signal $s(t)$ is given by:

$$s(t) \longrightarrow a^r \; s(a^{-1}(t-b)) \qquad (1)$$

where the real number r is determined by the physical dimension of the signal under consideration (change of unit of measurement).

Time-frequency representations of signals can be introduced as real sesquilinear forms of $s(t)$ which transform like:

$$P(t,f) \longrightarrow P'(t,f) = a^q \; P(a^{-1}(t-b), af) \qquad (2)$$

when the signal is transformed by (1). The real number q occuring in (2) characterizes the physical dimension of P. It is left free and has to be fitted to the required interpretation of P.

The most general time-frequency distribution transforming as (2) when the signal is transformed by (1) is given by [1]:

$$P_s(t,f) = \mid f \mid^{2r-q+2} \int_0^\infty \int_0^\infty e^{2i\pi t f(v-v')} \; K(v,v')S(fv)S^*(fv') \, dv \, dv' \qquad (3)$$

where $S(f)$ is the Fourier transform of $s(t)$ and where the kernel $K(v,v')$ is a real symmetric function. This latter condition on the kernel is in fact introduced in order to ensure that the representation of a physical signal verify:

$$P(t,f) \equiv P(t,-f)$$

Formula (3) does not involve any interference between positive and negative fequency contents of the signal under study.

This situation allows to limit the application of (3) to tl positive frequency part of a signal (analytic signal) in o der to get directly the positive frequency part of its tim frequency representation. In the following we will assun that all signals can be represented by real functions of tin and the above simplification will systematically be used. terms of the frequency description of signals, the action the affine group is given by:

$$Z(f) \rightarrow Z_{a,b}(f) = a^{r+1} e^{-2i\pi bf} Z(af) \qquad ($$

and this leads to introduce the invariant scalar product:

$$(Z,Z') = \int_0^\infty Z(f) Z'^*(f) f^{2r+1} \, df \qquad ($$

on the space of analytic signals.

A special subclass of (3) corresponds to diagonal kerne [1] [2] [3] and is written on the half-plane $f > 0$:

$$P(t,f) = f^{2r-q+2} \int_{-\infty}^{\infty} e^{2i\pi t f(\lambda(u)-\lambda(-u))} \; Z(f\lambda(u))Z^*(f\lambda(-u))\mu(u) \, du \qquad ($$

where $\lambda(u)$ is a positive function possibly submitted to ext conditions for special purposes.

The general aspect of formula (6) can be changed by sin ple modifications of the notations. A particular rewritir has been done in [4] through the trivial change of notation

$$\lambda(u) - \lambda(-u) = \phi(u)$$

$$\lambda(u) + \lambda(-u) = \psi(u)$$

followed by a reparametrization defined by:

$$u = \phi^{-1}(n)$$

The scientific advantage of the alteration of the original formula (6) has however not be argued.

In the framework of (6), interesting distributions are associated with the family of functions λ given by [1] [5] :

$$\lambda_k(u) = \left(k\frac{e^{-u}-1}{e^{-ku}-1}\right)^{\frac{1}{k-1}} \quad (7)$$

where the index k of the family is a real number. The particular functions corresponding to $k = 0$ and to $k = 1$ are defined by continuity with respect to the k variable. For $k \leq 0$, all corresponding distributions have remarkable localization properties. In this sub-family the case $k = 0$ stands out as the corresponding distribution looks like a multi-purpose tool. The object of the following is to give a straightforward derivation of this distribution as a member of the class (6). This is achieved after a systematic study of the properties of localization and unitarity. Illustration of the use of the distribution is given through radar imaging applications.

2 Localization

The concept of localization is basically connected to the transformation group of observers which, in the present case, is the affine group of clock changes. The study of the transformation properties of the signals under this group leads to an axiomatic introduction of localization. If an observer finds a signal $\mathcal{T}_{t_0}(f)$ localized in time at t_0 or a signal $\mathcal{F}_{f_0}(f)$ localized in frequency at f_0, then a different observer using a different time obtained from the first by dilation a and translation b must find the same signals localized respectively at at_0+b and $a^{-1}f_0$. Since the transformation properties of signals in clock changes are known, the localization constraints finally read:

$$\mathcal{T}_{at_0+b}(f) = a^{r+1}e^{-2i\pi bf}\mathcal{T}_{t_0}(af) \quad (8)$$

$$e^{-2i\pi bf_0}\mathcal{F}_{a^{-1}f_0}(f) = a^{r+1}e^{-2i\pi bf}\mathcal{F}_{f_0}(af) \quad (9)$$

Remark that the localization conditions depend on the parameter r, i.e. on the dimension of the signal.

To solve (8), we take its derivative with respect to b and set $a = 1, b = 0$ in the result. This gives:

$$\frac{d\mathcal{T}_{t_0}(f)}{dt_0} = -2i\pi f\mathcal{T}_{t_0} \quad (10)$$

whose solution is $\mathcal{T}_{t_0} = K(f)e^{-2i\pi ft_0}$ with K an arbitrary function.

Then take the derivative of (8) with respect to a and set again $a = 1, b = 0$:

$$t_0\frac{d\mathcal{T}_{t_0}(f)}{dt_0} = (r+1)\mathcal{T}_{t_0}(f) + f\frac{d\mathcal{T}_{t_0}(f)}{df} \quad (11)$$

Substituting the above form of $\mathcal{T}_{t_0}$ in (11) and solving the equations yields the form of the localized signal in time:

$$\mathcal{T}_{t_0}(f) = f^{-r-1}e^{-2i\pi ft_0} \quad (12)$$

In the same way, solving (9) leads to the localized signal in frequency:

$$\mathcal{F}_{f_0}(f) = f^{-r}\delta(f-f_0) \quad (13)$$

Expressions (12) and (13) are defined up to a complex multiplicative factor.

A satisfactory representation of these signals in the time-frequency plane must be of the general form:

$$P_{t_0}(t,f) = \tau(t,f)\delta(t-t_0)\ ;\quad P_{f_0}(t,f) = \phi(t,f)\delta(f-f_0)$$

where the arbitrary functions τ and ϕ will again be determined by covariance arguments. The counterparts of conditions (8) and (9) are now:

$$P_{at_0+b}(t,f) = a^q P_{t_0}(a^{-1}(t-b), af) \quad (14)$$

$$P_{a^{-1}f_0}(t,f) = a^q P_{f_0}(a^{-1}(t-b), af) \quad (15)$$

Here again the dimensional factor q shows up in the localization constraints. The solution to these equations is obtained as above and reads, up to an arbitrary constant:

$$P_{t_0}(t,f) = f^{-q-1}\delta(t-t_0) \quad (16)$$

$$P_{f_0}(t,f) = f^{1-q}\delta(f-f_0) \quad (17)$$

Now that the concept of localized signal is defined both in f-space and in the time-frequency domain, we enquire whether any of the affine representations introduced earlier (6) performs a good localization. The results are as follows:

- *frequency localization*

The signal $\mathcal{F}_{f_0}$ in (13) is represented by P_{f_0} given in (17) provided the function μ in (6) is such that $\mu(0) = 1$.

- *time localization*

The representation (6) of signals $\mathcal{T}_{t_0}$ defined by (12) is of the form (16) provided the following conditions are satisfied:

(i) $u \longrightarrow (\lambda(u) - \lambda(-u))$ is a one-to-one mapping from R to R.

(ii) the function μ is given in terms of λ by the expression:

$$\mu(u) = (\lambda(u)\lambda(-u))^{r+1} \mid \frac{d}{du}(\lambda(u) - \lambda(-u)) \mid \quad (18)$$

$$\equiv \mu_L(u)$$

We will now restrict our attention to this case and call the corresponding function $P_L(t,f)$ a localized affine distribution.

Beside general properties previously mentioned [1], [3], the class of distributions P_L defined by (6) and (18) allows an interesting time-frequency interpretation of the filtering operation. To develop this point we introduce the relation:

$$Z_T(f) = T(f)Z(f) \quad (19)$$

where the functions $Z(f)$, $T(f)$ and $Z_T(f)$ denote respectively a given signal, a filter and the associated output signal. The dimensional diagnosis of the relation (19) leads to note that if $Z(f)$ transforms with index r (cf relation (4)) and if $T(f)$ transforms with index r_T, then $Z_T(f)$ transforms with the index r' given by

$$r' = r + r_T + 1 \quad (20)$$

In classical applications where the input and output signals are of the same physical dimension, the index of the filter must be equal to -1. In the general case of transducers, however, the index r_T may take any real value.

The relation (19) has a direct counterpart in the time-frequency plane. To show this, we choose a special representation P_L in the class ((6), (18)) corresponding to a given function $\lambda(u)$, but we do not fix the dimensional indices. This choice permits to associate the distributions P_{Z_T}, P_T and P_Z of respective indices q', q_T and q_Z with the signals of relation (19). A direct calculation then leads to the formula:

$$P_{Z_T}(t,f) = P_T(t,f) \underset{t}{*} P_Z(t,f) \tag{21}$$

(convolution in time only) provided that the dimensional indices are connected by the relation

$$q' = q_T + q_Z + 1$$

In fact formula (21) is the affine counterpart of a well-known formula relative to the Wigner-Ville function.

3 Unitarity

The constraint of unitarity or "Moyal property" is expressed by the dimensionless relation:

$$\int_{R\times R^+} P_1(t,f)P_2(t,f)f^{2q}\, dt\, df \equiv \mid (Z_1, Z_2) \mid^2 \tag{22}$$

where the r.h.s. involves the invariant scalar product (5) and where P_1, P_2 are the distributions corresponding to signals Z_1, Z_2 respectively. Distributions of the diagonal form (6) may verify this identity provided that a definite relation exists between their arbitrary functions λ and μ. A direct computation shows that this relation is ensured by substituting to the function μ the functional:

$$\mu_U(u) = (\lambda(u)\lambda(-u))^{r+1/2} \times \tag{23}$$
$$(\mid \lambda'(u) + \lambda'(-u) \mid\mid \lambda(u)\lambda'(-u) + \lambda(-u)\lambda'(u) \mid)^{1/2}$$

As a result, it appears that there exists a large family of distributions ensuring (22) in the same way that (18) introduced a large family of localizable distributions. This situation opens the question of the existence of a distribution which would combine the two properties. Such distributions are characterized by the identification $\mu_L \equiv \mu_U$ which, by use of (18) and (23), leads to the equation:

$$\frac{d}{du}\left(\lambda(u) - \lambda(-u)\right) = \frac{d}{du} \ln\left(\frac{\lambda(u)}{\lambda(-u)}\right) \tag{24}$$

Integrating this equation with the condition $\lambda(0) = 1$ and setting $\lambda(u) - \lambda(-u) = V(u)$, we find the unique solution:

$$\lambda(u) = \frac{V(u)e^{V(u)}}{e^{V(u)} - 1} \quad ; \quad \lambda(-u) = \frac{V(u)}{e^{V(u)} - 1}$$

Changing variables from u to $\tilde{u} = V(u)$ and dropping the tilde, we finally get:

$$\tilde{P}(t,f) = f^{2r-q+2} \int_{-\infty}^{\infty} e^{2i\pi tfu} \times \tag{25}$$
$$Z\left(f\frac{ue^{u/2}}{2\sinh u/2}\right) Z^*\left(f\frac{ue^{-u/2}}{2\sinh u/2}\right)\left(\frac{u}{2\sinh u/2}\right)^{2r+2} du$$

This function is the closest counterpart of Wigner's function for the case of the affine group. The localization of chirps on straight lines which occured in the Wigner case is now replaced by the localization of definite signals ("Doppler tolerant") on hyperbolas and the role of the quadratic Fourier transform is now played by a particular Mellin transform. In fact, those remarks were at the starting point of the tomographic method [6] [7] used to obtain the form (25).

A fundamental property of representation $\tilde{P}$ is that it provides a direct geometric interpretation of the Mellin variable in the time-frequency plane [8]. This feature has allowed the formulation of a sampling theorem for the discrete Mellin transform of an analytic signal $Z(f)$. A fast Mellin transformation has then been developped and applied to the computation of $\tilde{P}$ as well as of other expressions involving dilations (e.g. wide-band ambiguity functions and wavelet coefficients [9] [8] [10]).

The easy implementation of the both localized and unitary affine distribution $\tilde{P}$ together with its many properties make it a good candidate for the analysis of wide-band signals in many situations. Here we shortly describe two different applications in radar imaging.

4 Delay-Doppler imaging of stochastic targets

An interesting formulation of the delay-Doppler imaging of random targets has been obtained in [11] in terms of the Wigner-Ville functions of the emitted and reflected signals. We rapidly show how this formulation can be extended in the wide-band case by using the time-frequency distribution (25).

Consider an incoming signal Z_E on a target made up of point scatterers. After reflection on a particular bright point of the target, the signal will have experienced a translation b in time and a Doppler dilation a in frequency related respectively to the position and velocity of that particular scatterer. Thus the received signal Z_R is of the form:

$$Z_R(f) = D\, a^{r+1} e^{-2i\pi fb} Z_E(af)$$

where the real number r depends on the physical nature of the signal (normalization) and the reflection coefficient D is specific of the target element.

Suppose now that the whole target can be characterized by a function $D(a,b)$ in such a way that the total received signal is given by:

$$Z_R(f) = \int_0^{\infty} \int_{-\infty}^{\infty} D(a,b) a^{r+1} e^{-2i\pi bf} Z_E(af)\, da db \tag{26}$$

Moreover, we assume that the function D is random and can be modeled by a stochastic process $D(a,b;\omega)$ whose covariance is :

$$E\left(D(a,b;\omega)D(a',b';\omega)\right) = \sigma(a,b)\delta(a-a')\delta(b-b') \tag{27}$$

This latter hypothesis simply expresses the stochastic independence of the elementary reflectors.

The affine distribution P_R of signal Z_R defined by (26) is computed using (25) and its expectation value is taken.

Taking into account (27), we obtain the result:

$$E(P_R(t,f)) = \int_{R\times R^+} \sigma(a,b) a^q P_E(a^{-1}(t-b), af)\, da\, db$$

where P_E is the affine distribution corresponding to Z_E. This has the form of a convolution on the affine group and can be inverted using the Fourier transformation on this group. It is just an ordinary convolution in time and a kind of multiplicative convolution in frequency. The interesting point is that $\sigma(a,b)$ can be recovered from the knowledge of P_R and P_E only. This conclusion stays in the line of the result obtained in the narrow-band case [11].

5 Microwave imaging in the laboratory

The study of a radar target in laboratory is founded on the measurement of its backscattering coefficient $H(f)$ which is a function of the illuminating frequency. However various definitions of this function can be given, depending on the physical context. In all cases the impinging wave is supposed to be plane and differences concern only the return wave which can be assimilated either to a plane, a cylindrical or a spherical wave. In the first case the backscattering coefficient is simply defined as the ratio

$$H_P(f) = E_{out}/E_{in} \qquad (28)$$

where E_{in} and E_{out} are respectively the incident and reflected fields (for definite polarizations). In the two other cases $H(f)$ is defined by the limits:

$$H_C(f) = lim_{R\to\infty}\sqrt{2\pi R}\,(E_{out}/E_{in}) \qquad (29)$$
$$H_S(f) = lim_{R\to\infty}\sqrt{4\pi R^2}\,(E_{out}/E_{in}) \qquad (30)$$

where R characterizes the distance between the radar and the target. According to the case, the square modulus of $H(f)$ (cross-section) is a scalar, a length or a surface.

An image of the target is defined as a repartition $\mathcal{I}(x,f)$ of localized bright points reflecting selectively. An explicit formulation of this description can be founded on the study of the effects of a modification of the basic model by displacing the points and changing their colors in the transformation:

$$(x,f) \longrightarrow (ax+b, a^{-1}f)$$

The representations of this deformation on the functions $H(f)$ and $\mathcal{I}(x,f)$ are obtained by a classical technique of physical similarity. We obtain:

$$H(f) \rightarrow H'(f) = a^{r+1}e^{-4i\pi f b/c}H(af) \qquad (31)$$
$$\mathcal{I}(x,f) \rightarrow \mathcal{I}'(x,f) = a^{2r+1}\mathcal{I}(a^{-1}(x-b), af) \qquad (32)$$

where the parameter r takes the values $(-1,-1/2,0)$ depending respectively on the definitions (28), (29) or (30) of the function H. The dimensional exponent in (32) is determined by the constraint

$$\int_{-\infty}^{\infty} \mathcal{I}(x,f)\, dx = | H(f) |^2$$

The above developments show the identity of the two prc lems of time-frequency representation and radar imaging a: lead to write:

$$\mathcal{I}(x,f) \equiv (2/c)\tilde{P}_H(2x/c, f)$$

$\tilde{P}$ is given by (25) with $q = 2r+1$ and r taking the sa value as in (31)-(32).

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