

**DISCRETE WAVELET TRANSFORMS:
THE RELATIONSHIP OF THE 'A TROUS AND MALLAT ALGORITHMS**

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RÉSUMÉ

L'objet de ce papier est de clarifier les rapports entre transformées d'ondelettes discrètes et continues. Plus précisément, on s'intéresse aux liens entre deux implémentations distinctes de la transformée d'ondelettes: l'algorithme à trous et la décomposition multirésolution de Mallat. On remarque que ces algorithmes sont tous deux des cas particuliers d'une unique structure banc de filtres, appelée transformée d'ondelettes discrète, dont le comportement dépend du choix des filtres. Dans cette approche l'algorithme à trous se présente comme un algorithme multirésolution non orthogonal pour lequel la transformée d'ondelettes discrète est exacte. Un cadre systématique est proposé pour cette transformée et des conditions pour lesquelles elle permet le calcul exact de la transformée d'ondelettes continue sont établies.

ABSTRACT

In a general sense this paper represents an effort to clarify the relationship of discrete and continuous wavelet transforms. More narrowly, it focuses on bringing together two separately motivated implementations of the wavelet transform, the *algorithme à trous* and Mallat's multiresolution decomposition. It is observed that these algorithms are both special cases of a single filter bank structure, the discrete wavelet transform, the behavior of which is governed by one's choice of filters. In fact, the *à trous* algorithm, originally devised as a computationally efficient implementation, is more properly viewed as a nonorthogonal multiresolution algorithm for which the discrete wavelet transform is exact. A systematic framework for the discrete wavelet transform is provided, and conditions are derived under which it computes the continuous wavelet transform exactly.

I. INTRODUCTION

Wavelets are rapidly finding application as a tool for the analysis of nonstationary signals [1]. However, with the notable exception of orthonormal wavelets ([2], [3]), very little literature has been devoted to linking discrete implementations to the continuous transform. As in the case of the discrete Fourier transform, these implementations (or filter banks) can, and should, stand on their own as abstract decompositions of discrete time series. Their wide sweeping significance, however, lies in their interpretation as wavelet transforms. In a general sense, this paper represents an effort to clarify the relationship of discrete and continuous wavelet transforms. More narrowly, it focuses on bringing together two separately motivated implementations of the wavelet transform. One of them, the *algorithme à trous* for nonorthogonal wavelets ([4], [5]), was developed for music synthesis ([1]) and is particularly suitable for signal processing. The other, the multiresolution approach of S. Mallat and Y. Meyer, originally used in image processing, employs orthonormal wavelets [2] - [3].

A glance at these two algorithms suffices to reveal closely related structures. In fact, apart from the constraints on their filters, the decimated *à trous* ([5]) and Mallat algorithms are identical. We are thus led to examine the expanded family of algorithms encompassing both types of filters. The algorithms to be discussed all are filter bank structures (see Figure 1). Their only distinguishing feature is the choice of two finite length filters, a lowpass filter f and a bandpass filter g . The lowpass condition, expressed

more precisely as $\sum f_k = \sqrt{2}$, is necessary to the construction of a corresponding continuous wavelet function [3]. The bandpass requirement, while apparently not essential to all applications, ensures that finite energy signals lead to finite energy transforms ([6]). Under these conditions the filter bank output will be referred to as the Discrete Wavelet Transform (DWT), a terminology which will become clear in the course of the paper.

One class of DWT filter pairs are the Daubechies filters [3] which yield orthogonal wavelet decompositions and constitute, in more conventional terms, a QMF filter bank with perfect reconstruction. Another is that for which the lowpass filter satisfies the *à trous* condition $f_{2k} = \delta(k) / \sqrt{2}$. Such filters, which simply serve to interpolate every other point, correspond to a nonorthogonal

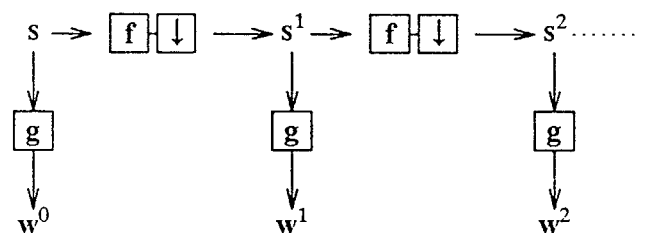


Figure 1. A wavelet filter bank structure. The down-arrow indicates decimation. The output of the transform is the family of signals w^i , forming the two parameter transform w_n^i in the scale-time plane. Following terminology to be introduced, w^i is the (decimated) discrete wavelet transform.



wavelet decomposition. If they are further restricted to be Lagrangian interpolators, they become the squares of the Daubechies filters [6], which is quite remarkable in consideration of the totally different derivations.

A fundamental question is when do these discrete implementations yield exact (i.e., sampled) versions of a continuous wavelet transform? Aside from regularity conditions relating to smoothness [3], we shall find that if f is *'a trous'*, then the DWT coincides with a continuous wavelet transform by a wavelet $g(t)$ whose samples $g(n)$ form the filter g . Even if f is not *'a trous'*, the algorithm is exact provided the signal lies in an appropriate subspace; however, in that instance, the corresponding wavelet depends on f as well as g . This is the situation in the orthonormal case where, moreover, the filter g is almost completely determined from f through the constraints of orthogonality.

II. TRANSFORM DEFINITIONS

The continuous wavelet transform of a signal $s(t)$ takes the form

$$W(a,b) = \frac{1}{\sqrt{a}} \int \bar{g}\left(\frac{t-b}{a}\right) s(t) dt \quad (1)$$

where g is the analyzing wavelet, a represents a time dilation, b a time translation, and the bar stands for complex conjugate. The normalization factor $1/\sqrt{a}$ is perhaps most effectively visualized as endowing $|W(a,b)|^2$ with units of power/hz. Certain weak "admissibility" conditions are usually required on $g(t)$ for it to be a candidate for an analyzing wavelet; namely, square integrability and $\int \frac{|\hat{g}(\omega)|^2}{|\omega|} d\omega < \infty$ where $\hat{g}(\omega)$ is the Fourier transform of $g(t)$. They insure that the transformation is a bounded invertible operator in the appropriate spaces ([3], [7]). If $\hat{g}(\omega)$ is differentiable, then it suffices that g be zero mean, i.e., that $\int g(t) dt = 0$.

We shall be exclusively concerned with discrete values for a and b . In particular, we assume that a has the form $a = 2^i$ where i is termed the *octave* of the transform. The integral (1) then yields a *wavelet series* $W(2^i, n) \triangleq \frac{1}{\sqrt{2^i}} \int \bar{g}\left(\frac{t-n}{2^i}\right) s(t) dt$. We remark that finite energy for the wavelet transform is not at all equivalent to finite energy for the wavelet series. It depends on the sampling grid as well as the function $g(t)$. Thus, the above admissibility condition is not necessarily appropriate in the discrete case and should be replaced with conditions on the relevant filters [6]. In the present paper, we shall take b to be a multiple of a

$$W(2^i, 2^i n) \triangleq \frac{1}{\sqrt{2^i}} \int \bar{g}\left(\frac{t}{2^i} - n\right) s(t) dt \quad (2)$$

A logical step in applying the theory to discrete signals is to discretize the integral in (2)

$$w(2^i, 2^i n) \triangleq \frac{1}{\sqrt{2^i}} \sum_k \bar{g}\left(\frac{k}{2^i} - n\right) s(k) \quad (3)$$

The sample rate has been set equal to one. As indicated by $2^i n$ on the left hand side, (3), as well as (2), are decimated wavelet transforms. Octave i is only output every 2^i samples. In this form the resulting algorithms will not be translation invariant. However, the invariance, which is lost by

decimation, is easily restored by separately filtering the even and odd sequences or by using the algorithm pictured in Figure 2 (c.f., [6]).

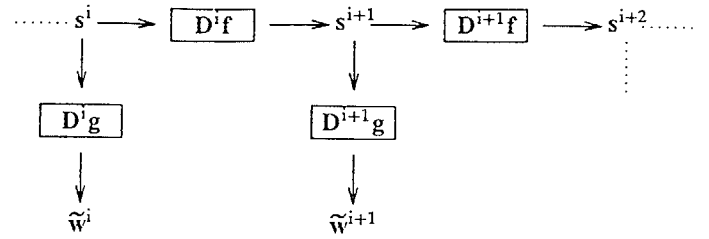


Figure 2. The (undecimated) discrete wavelet transform. The filters $D^i f$ are obtained from f by inserting $2^i - 1$ zeros between each pair of filter coefficients. The operation of filtering is understood to mean convolution.

Proceeding from (3), we shall arrive at the DWT of Fig. 1, namely,

$$[s^i]_n = \sum_j f_{2n-j} [s^{i-1}]_j \quad (4)$$

$$[w^i]_n = \sum_j g_{n-j} [s^{i-1}]_j$$

where $[w^i]_n$ corresponds to $w(2^i, 2^i n)$ of equation (3) and s^0 is the original signal s . The mysterious appearance of the filter f in (4) will be unraveled in the derivation of the *'a trous'* algorithm in Section III. Finally, we shall come full circle in Section IV where, under quite general conditions, we show the existence of a function $g(t)$ with $g(n) = g_n^\dagger$, and such that the DWT acting on the sampled signal is exactly the sampled output of the continuous wavelet transform (i.e., of the wavelet series).

Note that (1), (2), (3) and (4) have an analogy in the Fourier transform, Fourier series, discretized z-transform, and the discrete Fourier transform (DFT). The Fourier transform of a continuous signal $s(t)$, $S(\omega) \triangleq \int e^{-i\omega t} s(t) dt$, is a function of the continuous variable ω . Restricting it to a discrete (one-dimensional) grid results in the coefficients of a Fourier series $S(2\pi m) = \int e^{-2\pi i m t} s(t) dt$, which in turn may be computed approximately by $s_z(2\pi m \Delta t) = \sum_k e^{2\pi i m k \Delta t} s(k \Delta t) \Delta t$, the z-transform of $s_n \triangleq s(n \Delta t)$ output at discrete points $e^{2\pi i m \Delta t}$. If $s(t)$ is band-limited and sampled at an appropriate rate, $\Delta t = 1/N$, then the above may be computed exactly using the DFT, $\hat{s}_m \triangleq \frac{1}{N} \sum_{k=1}^N e^{\frac{2\pi i m k}{N}} s_k$. These transforms correspond precisely to $W(a, b)$, $W(2^i, n)$, $w(2^i, n)$, and undecimated w_n^i . With wavelets, however, we have the additional difficulty of dealing with a whole class of functions $g(t)$ rather than simply $e^{i\omega t}$. Also complicating things are its two-parameter structure and the existence of decimated versions, which, due to their $2^i n$ dependency on i , play a distinguished role without analogy in the one-dimensional case.

III. THE 'A TROUS AND MALLAT ALGORITHMS

Notation

Signals and filters in bold face will be treated as vectors, in which case $*$ indicates discrete convolution and yields a vector. The symbol \dagger is used for the adjoint filter $(f^\dagger)_k = \bar{f}_{-k}$. Decimation, which appears as a down arrow

in Fig. 1, plays a pivotal role in all DWT algorithms and will be represented by a matrix $\Delta_{kj} \triangleq \delta(2k - j)$ where $\delta(j)$ is the Dirac delta function. Also of significance is its transpose, $D_{kj} \triangleq \delta(k - 2j)$, which dilates a vector by inserting zeros.

The 'a trous algorithm

We take the discretized wavelet series (3) as a starting point. The difficulty in implementing (3) is that, even for $g(t)$ of finite support, as i increases, \bar{g} must be sampled at progressively more points, creating a large computational burden. The solution posed by [4] is to approximate the values at nonintegral points through interpolation via a finite filter f^\dagger . The resulting recursion is highly efficient and may be implemented with the filter bank structure of Fig. 1.

The interpolation is perhaps best introduced with an example. Let f^\dagger be the filter (0.5, 1.0, 0.5). Then,

$$\sum_k f_{n-2k}^\dagger g(k) = \begin{cases} g(\frac{n}{2}) & n \text{ even} \\ \frac{1}{2} (g(\frac{n-1}{2}) + g(\frac{n+1}{2})) & n \text{ odd} \end{cases} \quad (5)$$

approximates a sampling of $g(t/2)$. With the help of the dilation operator D , this may be formalized as a general procedure for dyadic interpolation. The steps are illustrated in Figure 3. Let g be a filter defined by $g_n^\dagger \triangleq g(n)$. First we spread g^\dagger to provide space in which to put the interpolated values. The resulting filter is Dg^\dagger . Then we apply a filter f^\dagger which leaves the even points fixed and interpolates to get the odd points. This condition, that f be the identity on even points, is sufficiently important to warrant a separate definition: The lowpass filter f is said to be an 'a trous filter if it satisfies

$$f_{2k} = \delta(k)/\sqrt{2} \quad (6)$$

The result of the entire interpolation operation, as pictured in Figure 3, is thus

$$[f^\dagger * (Dg^\dagger)]_n = \sum_k f_{n-2k}^\dagger g(k) \approx \frac{1}{\sqrt{2}} g(n/2) \quad (7)$$

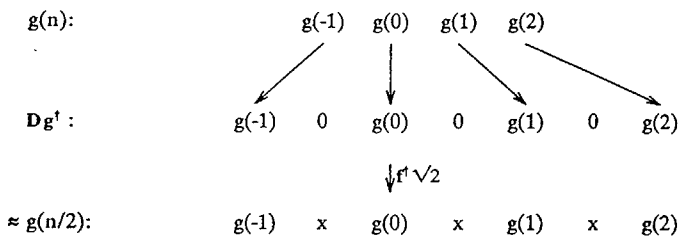


Figure 3. Diagram illustrating the dilation and interpolation of a function $g(t)$: $g(n/2) \approx \sqrt{2} f^\dagger * (Dg^\dagger)$.

Noting that $g(\frac{k}{2} - n) = g(\frac{k - 2n}{2})$ and inserting the approximation (7) into (3), we obtain

$$w(2, n) \approx \sum_{k,m} f_{k-2n-2m}^\dagger \bar{g}_m^\dagger s_k = [g * (\Delta(f * s))]_n \quad (8)$$

which is simply w_n^i of equation (4) with $i = 1$. Continuing inductively by replacing s in (8) with s^{i-1} , we find that $w(2^i, 2^n) \approx w_n^i$ for all i , where w_n^i is given by (4), which can

be rewritten

$$s^{i+1} = \Delta (f * s^i) \quad (9a)$$

$$w^i = g * s^i \quad (9b)$$

Except for decimation of the output, this is the 'a trous algorithm described in [4]. Thus, we see that the 'a trous algorithm is simply a DWT for which the filter f (an interpolator) satisfies condition (6) and the filter g is obtained by sampling an *a priori* wavelet function $g(t)$.

The Mallat Algorithm

With w_n^i replaced by w_{2n}^i , the recursions (9) coincide with those of the Mallat algorithm ([2], [3]). As is well known ([3]), in the orthonormal case, they provide the sampled output of a continuous wavelet transform; that is, w_n^i coincides with $W(2^i, 2^n)$ of (2) for suitable $g(t)$. More precisely, if the filters f and g satisfy

$$\sum_j \bar{f}_{2j-n} f_{2j-m} + \bar{g}_{2j-n} g_{2j-m} = \delta_{nm} \quad (10a)$$

$$\sum_j f_{2n-j} \bar{g}_{2m-j} = 0, \quad (10b)$$

plus certain regularity conditions, then

$$w_{2n}^i = \frac{1}{\sqrt{2^i}} \int s(t) \bar{g}(\frac{t}{2^i} - n) \quad (11)$$

provided that

$$s_k^0 = \int s(t) \bar{\phi}(t-k) \quad (12)$$

where ϕ (c.f., (14) below) is related to $g(t)$ and the filter g by

$$g(t) \triangleq \sum_k \sqrt{2} \bar{g}_{-k} \phi(2t - k) \quad (13)$$

It is extremely important to note that, although the discrete algorithm (4) is an exact computation of (11), the sampled signal must lie in an appropriate subspace (i.e., (12)), and the relationship between $g(t)$ and the filter g is relatively complex, i.e., $g_n \neq g(n)$.

IV. THE DWT AS AN EXACT WAVELET TRANSFORM

Regardless of the filters employed, one can, of course, perform the recursions (9) on the sampled signal s . Moreover, provided that f is lowpass and g bandpass, the procedure may be interpreted physically as a bank of proportional bandwidth filters [1], [6]. In the present section, we examine the mathematical significance of relaxing the filter constraints (6) and (10). Our goal will be to relate the more general filter bank to the continuous wavelet transform, thus, in a sense, justifying the term DWT. In this endeavor, the major questions which we shall address are: for what functions $g(t)$ is the recursion (9) an exact implementation of (3) and for which $g(t)$ and $s(t)$ do (2) and (3) coincide? The general answer is that we are able to construct such a $g(t)$ provided the discretized signal lies in the subspace dictated by (12). A somewhat surprising result is that it is necessary and sufficient for f to be 'a trous for condition (12) to be dropped. Due to a lack of space only the results will be presented. The reader is referred to [6] for a complete treatment including proofs.



The discrete algorithm is specified by a signal $s(t)$ and two discrete filters \mathbf{f} and \mathbf{g} . The wavelet transform \mathbf{w}^i is then determined from (9) and the initial conditions on s^0 . Our approach in relating this recursion to a continuous wavelet transform is to mimic the standard construction of orthonormal wavelets [3]. More precisely, we construct a scaling function $\phi(t)$ with Fourier transform

$$\hat{\phi}(\omega) \triangleq \prod_{j=1}^{\infty} \left(\frac{1}{\sqrt{2}} \bar{f}_z\left(\frac{\omega}{2^j}\right) \right) \quad (14)$$

where $\bar{f}_z(\omega) = (\mathbf{f}^\dagger)_z(\omega)$ is the z-transform of \mathbf{f}^\dagger . (Sufficient conditions on \mathbf{f} for the existence, boundedness, and smoothness of ϕ may be found in [3].) Note that for (14) to converge to a nonzero function, the factors must approach one; i.e., $f_z(0) = 1$, which implies $\sum_k f_k = \sqrt{2}$.

The obvious choice for s^0 is

$$s_n^0 \triangleq s(n) \quad (15a)$$

However, we shall also consider

$$s_n^0 \triangleq \sum_k \phi(k-n) s(k) \quad (15b)$$

which relates to the discretized wavelet series $w(2^i, 2^i n)$, and

$$s_n^0 \triangleq \int \phi(t-n) s(t) dt \quad (15c)$$

which corresponds to the sampled WT (wavelet series). For a given \mathbf{g} , we shall construct a continuous function $g(t)$ such that the DWT of equation (9) is an exact implementation of the discretized wavelet series under (15b) and of the wavelet transform under (15c).

Define $g(t)$ and $g_n^i(t)$ by

$$g(t) \triangleq \sum_k \phi(t+k) \bar{g}_k = \sum_k \phi(t-k) g_k^\dagger \quad (16)$$

$$g_n^i(t) \triangleq \frac{1}{\sqrt{2^i}} g\left(\frac{t}{2^i} - n\right) \quad (17)$$

The 'a trous' condition (6), $f_{2k} = \delta_{k0}/\sqrt{2}$, plays a central role as a consequence of the following theorem [6]:

$$\mathbf{f} \text{ is an 'a trous' filter} \iff \phi(n) = \delta_{n0} \quad (18)$$

This theorem implies that if \mathbf{f} is an 'a trous' filter, then $g(t)$ defined by (16) satisfies $g(n) = g_n^\dagger$. We proceed to give a summary of exactness results; i.e., conditions under which the transforms $W(2^i, 2^i n)$ (continuous signal and continuous wavelet) and $w(2^i, 2^i n)$ (discrete signal and continuous wavelet) are computed exactly by the discrete wavelet transform w_n^i (discrete signal and discrete wavelet filters). The proofs may be found in [6].

Exactness

Given discrete filters \mathbf{f} and \mathbf{g} such that $\phi(t)$ of (14) is well defined, define $g(t)$ and $g_n^i(t)$ by (16) - (17) with corresponding sampled wavelet transform (wavelet series)

$$W(2^i, 2^i n) \triangleq \int \bar{g}_n^i(t) s(t) dt \quad (19)$$

and discretized wavelet series

$$w(2^i, 2^i n) \triangleq \sum_k \bar{g}_n^i(k) s(k) \quad (20)$$

Let $\phi_n \cdot s$ stand for the scalar product $\sum_k \phi_n(k) s_k$, and $\phi_n(t) \cdot s(t)$ for the L^2 scalar product $\int \phi_n(t) s(t)$. Then:

$$\mathbf{f} \text{ is 'a trous'} \implies g(n) = g_n$$

For \mathbf{s} discrete:

$$\begin{aligned} s_n^0 = \phi_n \cdot s &\implies w(2^i, 2^i n) = w_n^i \\ s^0 = \mathbf{s} \text{ and } \mathbf{f} \text{ is 'a trous'} &\implies w(2^i, 2^i n) = w_n^i \end{aligned}$$

For $\mathbf{s}(t)$ continuous:

$$s_n^0 = \phi_n(t) \cdot s(t) \implies W(2^i, 2^i n) = w_n^i$$

Restrictions on the filter \mathbf{g}

Restrictions on the filter \mathbf{g} relate to finite energy and admissability, which, as mentioned in Section II, differ from the continuous case. Constrained by space, we limit ourselves to the following result from [6]: Let $\|\tilde{s}^\infty\|^2 \triangleq \limsup_{i \rightarrow \infty} \|s^i\|^2 / 2^i$. A sufficient condition for the undecimated DFT \tilde{w} and its inverse to satisfy

$$A \|s\|^2 \leq \sum_i \frac{1}{2^i} \|\tilde{w}^i\|^2 + \|\tilde{s}^\infty\|^2 \leq B \|s\|^2 \quad (21)$$

(i.e., to be bounded in an appropriate metric) is that, for all ω , $|f_z(\omega)|^2 \leq 2$ and

$$0 < A \leq \frac{\frac{1}{2} |g_z(\omega)|^2}{1 - \frac{1}{2} |f_z(\omega)|^2} \leq B < \infty \quad (22)$$

Note that a necessary condition for (22) to hold is $|f_z(\omega)| = 2 \iff g_z(\omega) = 0$, which is very similar to the continuous admissability condition.

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