

Ambiguity Function for Random Signals

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RÉSUMÉ

Cet article étudie la structure de ambiguïté de systèmes de localisation passive en milieux avec trajets multiples. Une définition basée en notions géométriques et de la Théorie de l'Information est appliquée à l'analyse de deux situations dont l'étude n'était pas possible avec les définitions antérieures. Dans la première, l'énergie du signal reçu dépend de la localisation de la source. On conclut que cette variabilité permet une réduction de l'ambiguïté. Dans la seconde, le spectre de la source n'est pas connu, ce qui implique une dégradation de la performance globale du système. Finalement, on étudie l'importance, du point de vue de l'ambiguïté, d'une modélisation complète des observations.

ABSTRACT

We study the ambiguity structure of passive location systems in multipath environments. A definition based on geometric/Information Theoretic concepts is applied to two situations which study was not possible with previously existing global analysis tools. In the first, the source spectrum is known, but the received signal component's energy depends on the source location. We see that this additional variability of the data results in a decrease of the model's ambiguity. In the second, the source spectrum is not known, resulting in a degradation of the global performance. Finally, we conclude about the improvement on global performance that a complete modeling (both temporal and spatial, as opposed to purely spatial) of the observed wavefield can provide.

1 Introduction

Tracking and location systems are being used in increasingly complex environments, such as multiple radiating sources and multipath channels. However, the fast development of signal processing algorithms for these non-trivial situations has not been followed by a corresponding development of global analysis tools. The classical ambiguity function of Woodward [10] implicitly admits a set of restrictive assumptions. Among others: exact knowledge of the statistical description of the incoming data for each possible value of the source location and single source scenarios. Generalizations of Woodward's ambiguity have been presented [9, 3] that allow for the consideration of more general channel models and stochastic narrowband source signals. However, the two fundamental limitations referred to above are still present.

In [8, 7, 5], we presented a definition of ambiguity that does not suffer from the limitations previously mentioned, and thus can be used to analyze the global performance of passive location systems, with an arbitrary number of sources present. The new definition recovers Woodward's when the active narrowband RADAR problem is considered.

In this paper, we consider the effect of dropping two hypotheses underlying Woodward's definition: (i) knowl-

edge of the source spectrum; (ii) constant received power. We will show how the ambiguity structure deviates from the one predicted by the classical definition. Dropping hypothesis (i) is essential when studying passive systems; condition (ii) is incompatible with modeling of the multipath structure of the underwater acoustic channel. We apply the definition presented in [8, 7], and show that it captures effectively the significant features of each situation considered.

The paper is organized as follows. We begin with a brief presentation of the new ambiguity function, in section 2. Section 3 is devoted to the study of the impact of variable received signal power. We consider a zero-mean stationary Gaussian source signal, with known spectrum, propagating through a multipath channel. A numerical example illustrates the behaviour of the new definition. In section 4 we consider that the source spectrum is not known at the receiver. An analytic expression for the ambiguity function is presented and interpreted. The ambiguity function clearly reflects the degradation in performance due to uncertainty about the source spectrum. Finally, in section 5, the impact of modeling the multipath structure in the global performance is analysed. It is shown that the information coded in the temporal alignment of the received replicas can improve the discrimination power of the model.



2 Ambiguity

In this section we briefly present the definition of the ambiguity function proposed in [8, 7, 5]. Our definition of ambiguity is motivated by a geometric interpretation of Maximum Likelihood (ML) estimates for exponential families. It is based on the Kullback directed divergence between probability densities

$$I(p : q) \triangleq E_p \left\{ \ln \frac{p}{q} \right\} \quad (1)$$

where p, q are probability density functions (pdf's) and $E_p \{ \cdot \} = \int \cdot p(x) dx$. For exponential families, the following relation between the ML estimate $\hat{\theta}$ of the unknown deterministic parameter θ and the Kullback directed divergence holds:

$$\hat{\theta} = \arg \min_{\theta \in \Theta} I(\hat{p}(r) : p(r|\theta)) \quad (2)$$

where $\hat{p}(r)$ is determined from the observations r , for details see [2, 1]. The previous relation singles out the Kullback directed divergence as the basic discriminating measure in deciding the value of θ . Pairs of parameters θ_0, θ with $\theta_0 \neq \theta$ that yield a small value of $I(p(r|\theta_0) : p(r|\theta))$ are likely to yield erroneous estimates $\hat{\theta} = \theta$ when θ_0 is the true value of the parameter. Based on these arguments, we proposed [8, 7] the following definition of ambiguity function:

$$\mathcal{A}(\theta_0, \theta) = 1 - \frac{I(\theta_0 : \theta)}{I_{SUP}(\theta_0)} \quad (3)$$

where $I_{SUP}(\theta_0)$ is an upper bound on the Kullback divergence between $p(r|\theta_0)$ and the pdf conditioned on any other value of θ . To simplify the notation, we use $I(\theta_0 : \theta)$ for $I(p(r|\theta_0) : p(r|\theta))$. The definition above is suitable when there are no other unknown parameters besides θ . It can be shown [5] that it yields the classical definition of ambiguity function when the active narrowband RADAR problem is considered. In passive systems, the source spectrum is not known, implying that the conditional pdf $p(r|\theta)$ is not exactly known. In this case, to each value of θ corresponds a family of pdf's, allowing for all the possible source spectral characteristics. Denote this family (for fixed θ) by \mathcal{G}^θ , and define

$$\tilde{I}(\theta_0 : \theta)_{\gamma_0} \triangleq \min_{f \in \mathcal{G}^\theta} I(p(\theta_0, \gamma_0) : f) \quad (4)$$

and where γ_0 indicates the true source spectral parameters. The previous definition of ambiguity, (3), is extended to this case using this minimum value of the Kullback divergence:

$$\mathcal{A}(\theta_0, \theta)_{\gamma_0} \triangleq 1 - \frac{\tilde{I}(\theta_0 : \theta)_{\gamma_0}}{\tilde{I}_{SUP}(\theta_0)_{\gamma_0}}. \quad (5)$$

As in the previous case, $\tilde{I}_{SUP}(\theta_0)_{\gamma_0}$ denotes an upper bound, in this case on the minimum divergence \tilde{I} .

For zero mean stationary Gaussian signals, $\mathcal{A}(\theta_0, \theta)$ can be written in terms of the spectral density of the observations. Let $R_\theta(\omega)$ denote the spectral density of the observations, parameterized by the source location (θ). Then,

under mild conditions, [4, 8]

$$\mathcal{A}(\theta_0, \theta) = \frac{\tilde{I}_{SUP}(\theta_0) - \tilde{I}(\theta_0 : \theta)}{\tilde{I}_{SUP}(\theta_0)},$$

where $\tilde{I}(x : y)$ is defined by

$$\begin{aligned} \tilde{I}(y : x) &= \frac{1}{2} \int [\text{tr}[R_x(\lambda)R_y(\lambda)^{-1}] - K \\ &\quad - \ln \frac{|R_x(\lambda)|}{|R_y(\lambda)|}] d\lambda \end{aligned}$$

and K is the dimension of the observation vector. The previous expression identifies the Itakura-Saito distortion measure with the asymptotic value of the directed divergence.

3 Known Source Spectrum

We consider now the application of the previous definitions to the study of the ambiguity in locating a source of *known* power spectrum in a multipath channel. The observation's power spectrum is

$$R_\theta(\omega) = \mathcal{S}(\omega)(\omega)h_\theta(\omega)h_\theta(\omega)^H + \sigma^2(\omega)I_K$$

where we assumed that the observation noise is spatially incoherent, with known power density $\sigma^2(\omega)$. In the previous equation, $\mathcal{S}(\omega)$ (ω) is the source spectral density and $h_\theta(\omega)$ is the resultant vector, that describes the superposition of all the steering vectors corresponding to each replica received. Applying definition 3, the following expression is obtained, [8]

$$\begin{aligned} \mathcal{A}(\theta_0, \theta) &= 1 - \frac{1}{2C \int \text{SNR}(\omega)} \int [\text{SNR}(\omega) \|h_{\theta_0}(\omega)\|^2 \\ &\quad - \frac{\text{SNR}(\omega)^2}{1 + \text{SNR}(\omega) \|h_\theta(\omega)\|^2} |h_{\theta_0}(\omega)^H h_\theta(\omega)|^2 \\ &\quad - \frac{\text{SNR}(\omega)}{1 + \text{SNR}(\omega) \|h_\theta(\omega)\|^2} \|h_\theta(\omega)\|^2 \\ &\quad + \ln \frac{1 + \text{SNR}(\omega) \|h_\theta(\omega)\|^2}{1 + \text{SNR}(\omega) \|h_{\theta_0}(\omega)\|^2}] d\omega. \quad (6) \end{aligned}$$

where $\text{SNR}(\omega)$ denotes the source signal to noise ratio,

$$\text{SNR}(\omega) \triangleq \frac{\mathcal{S}(\omega)}{\sigma^2(\omega)},$$

and C is an upper bound on the value of $\|h_\theta(\omega)\|^2$.

To get insight into the meaning of the previous expression, we analyze its limit for large values of $\text{SNR}(\omega)$ for flat source and noise spectra. A few lines of calculus lead to

$$\begin{aligned} \lim_{\text{SNR}_0 \rightarrow \infty} \mathcal{A}(\theta_0, \theta) &= 1 + \\ &\quad \frac{1}{2W} \int \frac{\|h_{\theta_0}(\omega)\|^2}{C} \left[\frac{\mathcal{A}(\theta_0, \theta)^{(c)}}{\|h_{\theta_0}(\omega)\|^2 \|h_\theta(\omega)\|^2} - 1 \right] d\omega. \end{aligned}$$

where $\mathcal{A}(\theta_0, \theta)^{(c)}$ denotes the unnormalized equivalent of the classical ambiguity function for each processing frequency:

$$\mathcal{A}(\theta_0, \theta)^{(c)} \triangleq |h_{\theta_0}(\omega)^H h_\theta(\omega)|^2. \quad (7)$$

This expression shows that for sufficiently high signal to noise ratio the ambiguity structure is dependent only on the propagation/observation operators, and not on the source spectrum. Note also that the analogue of the classical ambiguity is weighted by the effectively received signal power, through the norm of the resultant vector at the true source position, $h_{\theta_0}(\omega)$.

Below, (Fig. 2), we show a plot of the ambiguity function for the simple configuration illustrated in Fig. 1. The

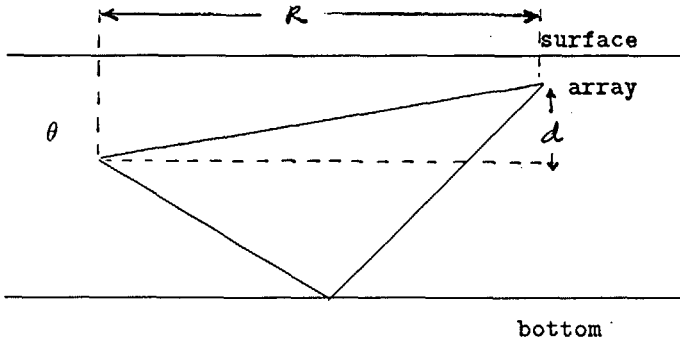


Figure 1: Geometry for Fig. 2.

sound velocity is constant, and equal to 1500 m/s. The 10 element receiving uniform linear array is vertical, with inter-sensor spacing 0.1 m. The source and noise spectra are constant over the bandwidth [100, 500].

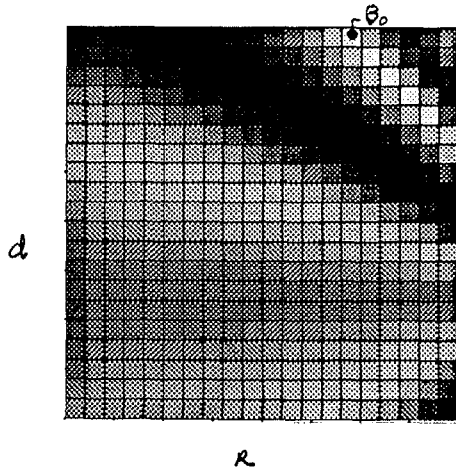


Figure 2: Density plot of the ambiguity.

While the ambiguity surface shows the existence of side-lobes of considerable height, oriented in an almost radial fashion, due to the lack of resolving power of the array for the frequencies of analysis, the surface predicted by the classical ambiguity is almost completely flat.

4 Unknown Source Spectrum

In this section we consider the same scenario as in the previous section, now for a source radiating a signal of *unknown* spectrum. In this case the probability density of the received data is dependent on the (unknown) source spectrum, and the general definition (5) must be used,

resulting in [8],

$$\mathcal{A}(\theta_0, \theta)_{S_0} = \int \frac{\text{SNR}(\omega)}{\int \text{SNR}(\omega) d\omega} \mathcal{A}(\theta_0, \theta)_{S_0}^{(c)} \quad (8)$$

$$- \frac{1}{\int \text{SNR}(\omega) d\omega} \ln \frac{1 + \text{SNR}(\omega) \mathcal{A}(\theta_0, \theta)_{S_0}^{(c)}}{1 + \text{SNR}(\omega)} d\omega$$

where $\text{SNR}(\omega)$ is the ratio of *received* signal to noise power,

$$\text{SNR}(\omega) \triangleq \frac{S_0(\omega) \|h_{\theta_0}(\omega)\|^2}{\sigma^2(\omega)}, \quad (9)$$

and $\mathcal{A}(\theta_0, \theta)_{S_0}^{(c)}$ is the analogue of the classical ambiguity function, i.e., the square of the cosine of the angle between the resultant vectors for the two values of source location.

$$\mathcal{A}(\theta_0, \theta)_{S_0}^{(c)} \triangleq \frac{|h_{\theta_0}(\omega)^H h_{\theta}(\omega)|^2}{\|h_{\theta}(\omega)\|^2 \|h_{\theta_0}(\omega)\|^2}.$$

Note that the presence of the logarithmic term sets a lower bound on the possible values of $\mathcal{A}(\theta_0, \theta)$:

$$\mathcal{A}(\theta_0, \theta) \geq \frac{1}{\int \text{SNR}(\omega)} \int \text{SNR}(\omega) \ln(1 + \text{SNR}(\omega)) d\omega.$$

This bound (see the definition of $\text{SNR}(\omega)$) depends on the *effectively received energy*.

5 Temporal Modeling

In this section we study the impact of a complete spatial/temporal modeling of the received wavefield, when compared to pure spatial modeling. We consider the passive situation, where the source spectrum is not known.

For the complete model, the ambiguity function is given by the eq. (8) presented in the previous section, that we denote here by $\mathcal{A}(\theta_0, \theta)^{sp/temp}$.

Consider the following decomposition of the resultant vector:

$$h_{\theta}(\omega) = D(\theta)b(\theta)$$

where $D(\theta)$ is the matrix of direction vectors for each wavefront incoming, and $b(\theta)$ is a complex vector that describes the relative delays and attenuations of the incoming paths at a reference sensor.

The matrix $D(\theta)$ describes the spatial structure of the received data, and $b(\theta)$ its temporal alignment. Pure spatial models use only the dependency of $D(\theta)$ on θ , considering b as an unknown deterministic vector. The simultaneous uncertainty about the source spectrum and the vector b imply that only the product $\sqrt{S(\omega)}b$ can be determined. Applying definition (5) yields [8]

$$\mathcal{A}(\theta_0, \theta)_{b_0}^{sp} = \int \frac{\text{SNR}(\omega)}{\int \text{SNR}(\omega) d\omega} \left[\mathcal{A}(\theta_0, \theta)_{b_0}^{(d)} - \frac{1}{\text{SNR}(\omega)} \ln \frac{1 + \text{SNR}(\omega) \mathcal{A}(\theta_0, \theta)_{b_0}^{(d)}}{1 + \text{SNR}(\omega)} \right] d\omega$$

where now

$$\mathcal{A}(\theta_0, \theta)_{b_0}^{(d)} \triangleq \frac{|\Pi_{\mathcal{H}(\theta)} D(\theta_0) b_0|^2}{\|D(\theta_0) b_0\|^2}$$



denotes the cosine of the angle between the received resultant vector $h_\theta(\omega)$ and its projection on the space generated by the columns of $D(\theta)$, $\mathcal{H}(\theta)$, and SNR(ω) has the same definition as in the previous section, (9).

Comparing the two expressions, we can conclude immediately that

$$\begin{aligned} \mathcal{A}(\theta_0, \theta)^{(d)} &\geq \mathcal{A}(\theta_0, \theta)^{(s)} \\ \implies \mathcal{A}(\theta_0, \theta)^{sp} &\geq \mathcal{A}(\theta_0, \theta)^{sp/temp} \end{aligned}$$

i.e., the ambiguity surface for the complete model is below the ambiguity surface for the spatial model.

Consider that the receiving array is able to perfectly distinguish the paths incoming from each source alone,

$$D(\theta)^H D(\theta) \simeq D(\theta_0)^H D(\theta_0) \simeq KI$$

but that there are pairs of paths incoming from the two sources that fall inside the resolution limits of the array, such that for some pair of sequences i_0, i (see Fig. 3)

$$D(\theta_0)^H D(\theta) \simeq K \sum_k e_{i_0(k)} e_{i(k)}$$

where e_i denotes the i -th Euclidean vector. With these

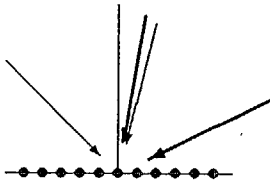


Figure 3: Geometry of arrivals.

assumptions, the ambiguity function can be written as [8]

$$\begin{aligned} \mathcal{A}(\theta_0, \theta)^{sp/temp} &= \frac{\int \text{SNR}(\omega)^* \overline{\mathcal{A}(\theta_0, \theta)^{(v)}}}{\int \text{SNR}(\omega)} \\ &- \frac{1}{\int \text{SNR}(\omega)} \int \ln \frac{1 + \text{SNR}(\omega)^*}{1 + \text{SNR}(\omega)} d\omega, \end{aligned}$$

where

$$\begin{aligned} \text{SNR}(\omega)^* &\triangleq \rho \text{SNR}(\omega) \\ \rho &\triangleq \frac{\|\check{b}_0\|^2 \|\check{b}\|^2}{\|b_0\|^2 \|b\|^2} \\ \mathcal{A}(\theta_0, \theta)^{(v)} &\triangleq \frac{|\check{b}_0^H \check{b}|^2}{\|\check{b}_0\|^2 \|\check{b}\|^2} \end{aligned}$$

and the r -dimensional vectors \check{b}_0, \check{b} are

$$\begin{aligned} \check{b}_0 &= [b_{i_0(1)} \cdots b_{i_0(r)}] \\ \check{b} &= [b_{i(1)} \cdots b_{i(r)}] \end{aligned}$$

We point out that $\overline{\mathcal{A}(\theta_0, \theta)^{(v)}}$ is the ambiguity function for a virtual array which resultant vector (\check{b}) is defined by the temporal delays between the unresolved received paths. Previously, the same notion of virtual array has been used

to explain the improvement in local performance (Cramér-Rao bound) when comparing complete and purely spatial modeling of the channel, see [6]. Finally, note that in the situation of very poor spatial resolution, the spatial model is completely ambiguous, and the ambiguity of the complete model is determined by the geometry of the virtual array. In this situation, the temporal modeling of the channel can greatly improve the global ambiguity picture. When there is almost perfect spatial resolution, the spatial model displays already a very low ambiguity, leaving no room for improvements, when the additional information on b is considered.

References

- [1] B. Effron. The geometry of the exponential families. *Ann. Statist.*, 6, pages 362–376, 1987.
- [2] Solomon Kullback. *Information Theory and Statistics*. Peter Smith, 1978.
- [3] José M. F. Moura. Passive systems theory with narrow-band and linear constraints: Part iii — spatial/temporal diversity. *IEEE J. of Oceanic Eng.*, pages 113–119, July 1979.
- [4] M. S. Pinsky. *Information and Information Stability of Random Variables and Processes*. Holden-Day, Inc., 1964.
- [5] M. João D. Rendas and José M. F. Moura. Ambiguity analysis in source localization with unknown signals. In *Int. Conf. on Acoustic, Speech and Signal Processing 91, Toronto, Canada*, May 1991.
- [6] M. João D. Rendas and José M. F. Moura. Cramer-Rao bounds for source location systems in multipath environments. *To appear in the IEEE Trans. on Acoustic, Speech and Signal Processing*, Dec. 1991.
- [7] M. João D. Rendas and José M. F. Moura. Ambiguity analysis in source location. *Submitted for publication*, May, 1991.
- [8] Maria João Rendas. *Erro e Ambiguidade em Sistemas de Localização*. PhD thesis, Instituto Superior Técnico, Lisboa, 1991.
- [9] H. L. Van Trees. *Detection, Estimation, and Modulation Theory, Part III*. John Wiley & Sons, Inc., 1971.
- [10] P. M. Woodward. *Probability and Information Theory, with Applications to RADAR*. Pergamon Press, 1953.