TREIZIÈME COLLOQUE GRETSI - JUAN-LES-PINS DU 16 AU 20 SEPTEMBRE 1991

ESTIMATION OF PARAMETERS OF NON—SINUSOIDAL SIGNALS BY
THE METHOD OF STRUCTURAL PROPERTIES — A SINC TEST CASE

Ilan DRUCKMANN

Dep. of Electrical Engineering, Ben Gurion University
P.O.Box 653, Beer Sheva, Israel

RÉSUMÉ

La méthode de propriétés de structure est appliquée à un exemple consistant de la mesure de la fréquence d'un signal sinc bruité.

ABSTRACT

The structural properties method can be used to estimate parameters of arbitrary signals. As a test case, the frequency estimation of a sinc is considered. Good results are obtained for short data records.

1. INTRODUCTION

Estimation of parameters (frequency, amplitude, phase) of multiple sinusoids imbedded in noise is a subject abundantly treated in signal processing literature, and a large number of methods and variants of methods have been suggested. On the other hand, little work, if any, has been done to deal with non-sinusoidal signals (see in this context [1],[2]). Clearly, the interest in multiple sinusoids is due to the ubiquitous presence of this kind of signals in physical systems, but one may wonder if, at least part of the reason for this interest lies in the fact that one knows how to deal with sinusoids, but not with other signals. Indeed, processing multiple sinusoids is equivalent to processing a linear, constant-coefficients differential equation, which is simply related to a linear, constant-coefficients difference equation, and this kind of equation is easy to deal with.

The structural properties approach [3] affords tackling the problem of parameter estimation in the case of arbitrary signals by using the relationships between the signal and its derivatives. These relationships express themselves as a Generating Differential Equation (GDE) for the signal under discussion, i.e. a differential equation which has the signal as a solution. These GDEs are time-dependent, or non-linear, or both, and their corresponding difference equations are not readily derived, as in the case of multiple sinusoids.

In this paper we investigate, as a test case, the frequency estimation of a sinc signal imbedded in a normally distributed white noise:

$$x(t) = \frac{\sin w(t - T/2)}{w(t - T/2)} + n(t), \qquad 0 \le t \le T \qquad (1)$$

where $w = 2\pi f$, and f is defined as the frequency.

We investigate by computer simulation the rms error and the bias of the error, and how they are affected by the sampling rate and the signal-to-noise ratio (SNR). These algorithms are highly dependent upon the differentiation method used. Two of them are compared here: an interpolating polynomial formula and an FFT based method.

2. A TIME INDEPENDENT GDE FOR THE SINC FUNCTION

A linear, but time-dependent, GDE for the sinc function is [3, p.164]:

$$t\dddot{x} + 2\ddot{x} + w^2 t x = 0. \qquad (2)$$

This GDE can already form the basis for a frequency estimator, but it is desirable to derive a new GDE which is time independent, in order not to be bothered by synchronization problems. By differentiating (1) and substituting t between them, we obtain the GDE:

$$w^2 x^2 + 2w(2x\dot{x} - \dot{x}^2) + 3\ddot{x}^2 - 2\dot{x}\dddot{x} = 0 \qquad (3)$$

which is time independent, but non-linear. (Clearly, there is a trade-off between linearity and time independence). Note that this GDE is invariant to time

delay or phase in the signal.

This GDE can be used for computing the frequency in the case there is no noise. The procedure consists of computing the first 3 derivatives and solving a quadratic algebraic equation in w. We derive next an algorithm for estimating the frequency in the presence of noise.

3. ALGORITHM FOR ESTIMATING THE FREQUENCY IN THE PRESENCE OF NOISE

If the signal is contaminated by noise, the GDE in (3), viewed as an operator on x, does not produces zero, but there is an error:

$$GDE(x,\dot{x},\ddot{x},\dddot{x}) = e \qquad (4)$$

where GDE denotes the GDE of (3). We shall minimize this error in a mean square sense during the time interval in which the signal is available. Squaring the equation (4), averaging over time, and differentiating with respect to w^2 produces the following algebraic equation:

$$au^3 + bu^2 + cu + d = 0 \qquad (5a)$$

where u denotes w^2, and:

$$a = \langle x^4 \rangle$$
$$b = 3 \langle x^2(2x\ddot{x} - \dot{x}^2) \rangle \qquad (5b)$$
$$c = \langle 11x^2\ddot{x}^2 + 2\dot{x}^4 - 2x^2\dot{x}\dddot{x} - 8x\dot{x}^2\ddot{x} \rangle$$
$$d = \langle (3\ddot{x}^2 - 2\dot{x}\dddot{x})(2x\ddot{x} - \dot{x}^2) \rangle$$

and $\langle \cdot \rangle$ denotes time averaging on interval T. Note that this equation can be solved analytically.

The algorithm for estimating the frequency is therefore as follows. Given samples of the contaminated signal (1) during the interval T:

1. Compute the first 3 derivatives by any method (on differentiation methods see next section).
2. Compute the coefficients (5b).
3. Solve the cubic equation (5a) in unknown u. It has been shown empirically that the valid solution is always given by the first of the three solution formulas (first in the sense of [4]), both in the case D < 0 and D > 0.

4. Compute the frequency = $\sqrt{u} / 2\pi$.

4. DIFFERENTIATION METHODS USED

In our simulation we have used 2 differentiation methods. The first one is based upon a Lagrange polynomial of degree 6, interpolating 7 adjacent samples of the signal. According to this method the derivative at time t[n] is estimated by:

$$\dot{x}_n = \frac{45(x_{n+1} - x_{n-1}) - 9(x_{n+2} - x_{n-2}) + x_{n+1} - x_{n-1}}{60}$$

Given NP samples during T, this method always computes the derivative at NP -6 points, each time operator (6) is used. In our case, 3 derivatives are required, therefore we must discard at the edges of the time interval 3·6 = 18 samples, which are not available for subsequent computation (averaging, etc.). This also sets a minimum for the number of samples required by this method: NP = 19. If averaging is required, for example for low SNR, then NP must be larger.

The second method is based on FFT. The FFT is performed on the data sequence, obtaining X[k], k= 0..NP/2. The differentiation is performed in the frequency domain by computing the new sequences: jw[k]X[k], $-w^2$[k]X[k], $-jw^3$[k]X[k], where w[k] =2πk/T, and transforming back to the time domain by IFFT. Note that only 2 FFT transformations are required. In this method NP must be a power of 2.

5. SIMULATIONS AND RESULTS

In the simulations, the following parameters and notations have been used. Without loss of generality, the frequency was normalized to f = 1, and therefore the unit of time 1/f was also normalized (1/f is the width of the main lobe of sinc).
T is the time interval, T=6 in all simulations.
NP is the number of samples during T.
N is the sampling rate, N=NP/T.
When the polynomial differentiation method was used, subsequent computations were made upon NP - 18 samples (discarding 9 samples at each edge).

When the FFT method was used, NP was 16 or 32, and all samples were used.

A normalized error was defined as:

$$e = (computed\ frequency - 1)/f \quad (but\ f=1)$$

Its bias is:

$$bias = (\Sigma\ e[j]) / J$$

where J is the number of experiments using different noise sequences, and summation is over j=1..J.
The normalized rms error is:

$$rmse = \sqrt{\Sigma\ e_j^2 / J}$$

The number of experiments was: J=400 for the polynomial method, J=120 for the FFT-based method.

In a first set of simulations, we have investigated the influence of the sampling rate N on the estimation error, at high SNR (40 dB) and low SNR (-5 dB), for both differentiation methods.

The results for the error bias is shown in Fig. 1, and for the rms error, in Fig.2 (logarithmic scale). For high SNR, the bias is very small (less than 0.01) for both methods, but the polynomial method requires N>5. Also, the bias of the FFT method is smaller (this is not apparent in the graph). At low SNR the bias is highly dependent on N. Small bias can be achieved only for special rates: in the vicinity of N=5 (polynomial method) or N=2.66 (FFT method).

Rms error: for high SNR it follows the behavior of the bias. For low SNR, the minimum error is attained at the same rates as the minimum bias. Moreover, the minimal error obtainable by both methods is roughly the same: 15%.

In the second set of simulations, we have investigated the influence of the SNR on the error, at fixed sampling rate. The sampling rate was chosen to be optimal for each method: N=5 (NP=30) for the polynomial method, N=2.66 (NP=16) for the FFT method. The results are seen in Fig. 3 (bias) and Fig. 4 (rms error). Both graphs exhibit similar behavior for the two methods. The bias is less than 10% always; for high SNR (> 20 dB) it is less than 1%. The FFT method exhibits a better performance at almost all SNR, except at very low (-5 to 0 dB).
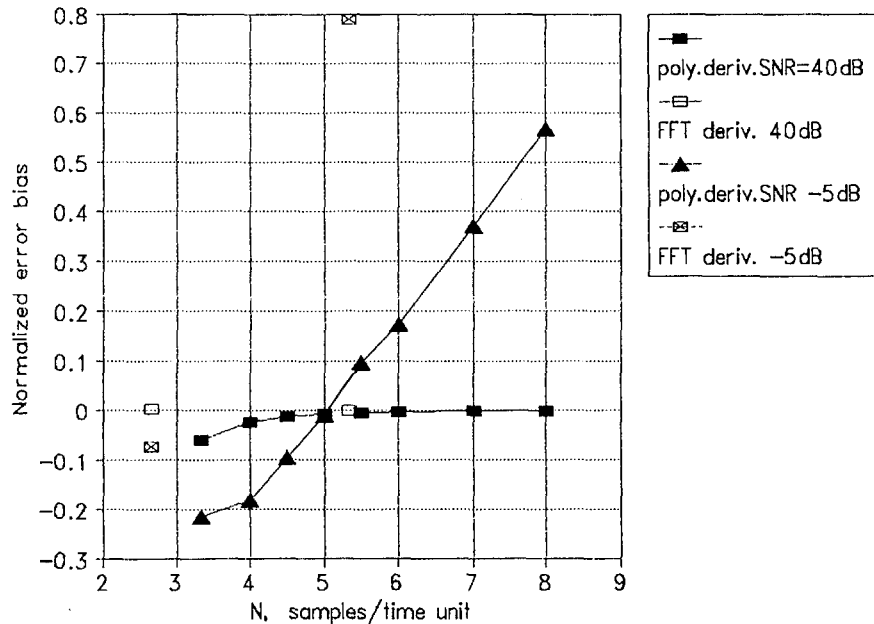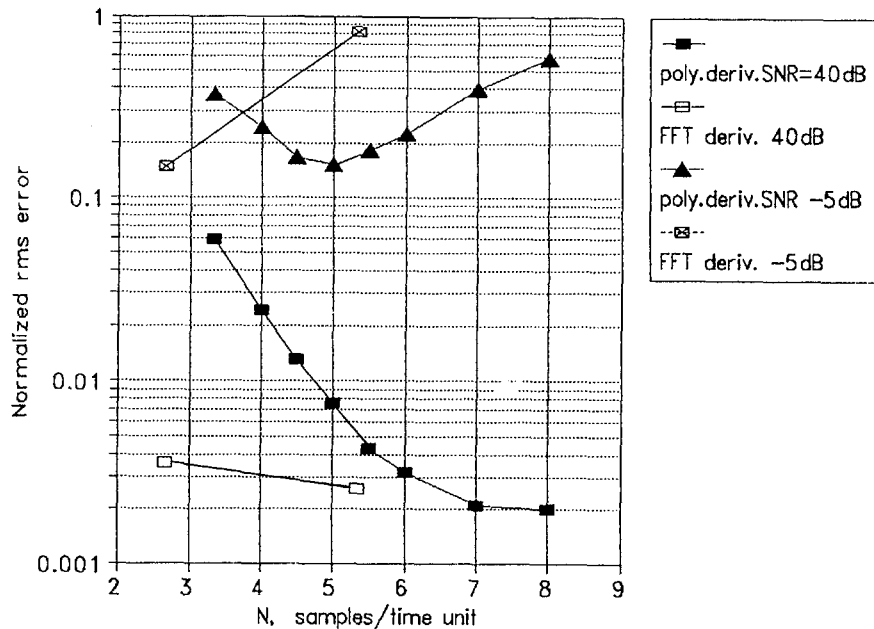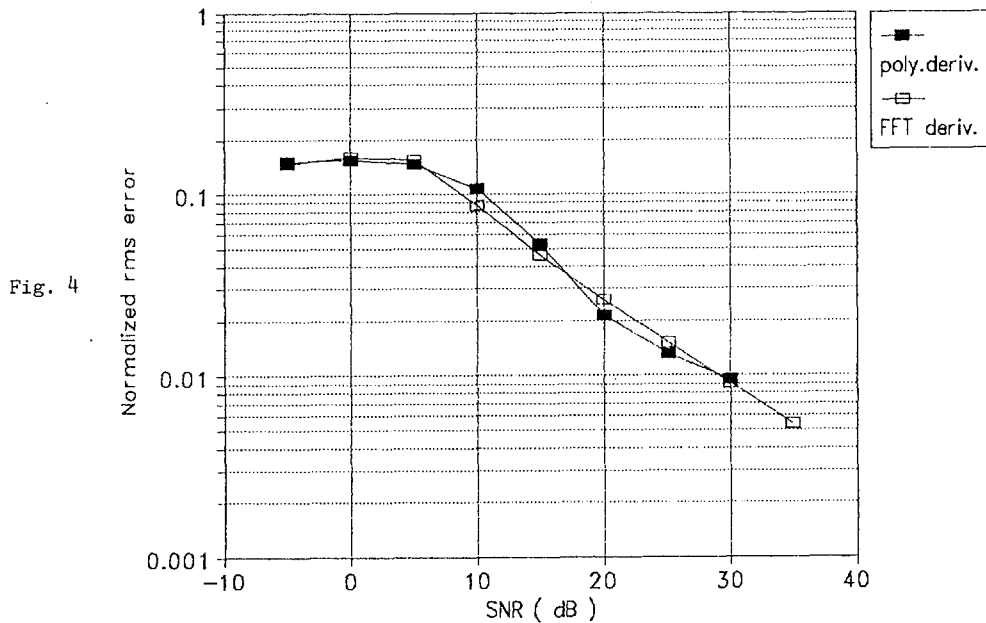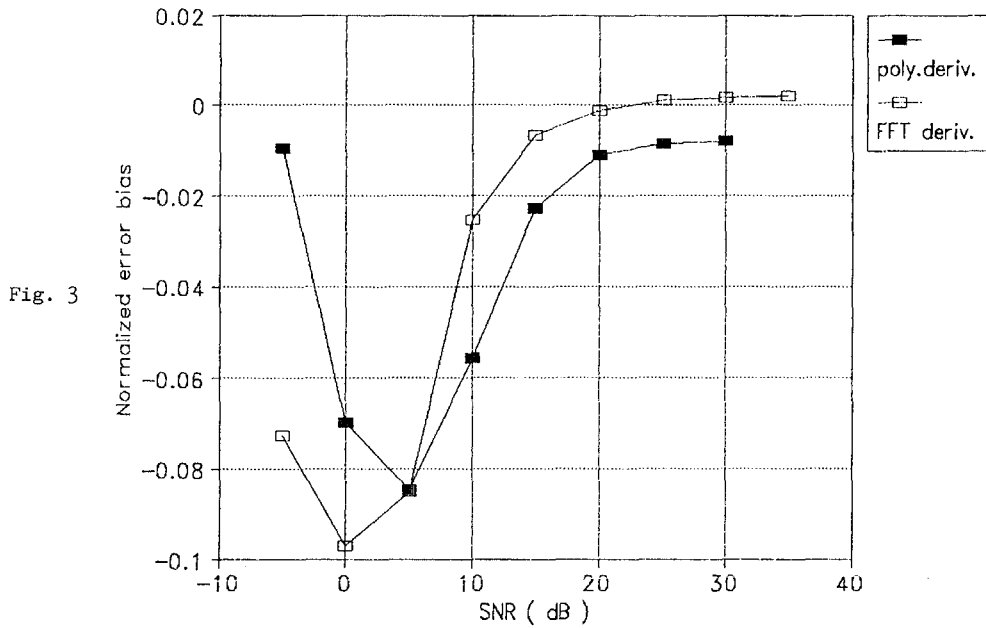

Fig. 1

Fig. 2

Fig. 3

Fig. 4

6. CONCLUSION

It has been shown that the method of structural properties can be used to estimate parameters of arbitrary signals imbedded in noise. Using high derivatives (up to 3rd derivative in this special case), even in presence of noise, is not an impediment for the method. It should be emphasized, that no special smoothing has been performed before or after differentiation, except for the averaging inherent in the algorithm (which arises from minimizing the mse). The test case has shown that short records of data (16 - 48 samples, a time window of 6 lobe widths of sinc) are sufficient for acceptable results: rms error of 15% is obtainable even at low SNR (-5 dB). The FFT method requires smaller sampling rates in general and performs better (less dependence on N) at high SNR (>10 dB). The method can be improved by using other

differentiation algorithms, and also can be applied to other signals as well.

REFERENCES

[1] B. Porat and B. Friedlander, "Adaptive detection of transient signals," IEEE Trans. ASSP, vol. T-ASSP 34, no.6, pp.1410-1418, Dec. 1986.

[2] A. Nehorai and P. Stoica, "Adaptive parameter estimation for a constrained low-pass system," IEEE Trans. Autom. Contr., vol.T-AC 33, no.1, pp.109-112, Jan. 1988.

[3] A.M. Zayezdny and I. Druckmann, "A new method of signal description and its applications to signal processing," Signal Processing, vol.22, no.2, pp.153-178, Feb. 1991.

[4] M.R. Spiegel, Mathematical Handbook, Schaum series, Mc.Graw-Hill, 1968, p.32.