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DATA COMPRESSION USING THE VARIABLE COEFFICIENTS OF A DIFFERENTIAL EQUATION - THEORY AND EXPERIMENTAL RESULTS

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RÉSUMÉ

Une équation différentielle (ED) qui a le signal x comme sa solution est la ED de génération (EDG) de x. Des EDG dont les coefficients varient plus lentement que x peuvent servir pour la compression de données. On presente deux EDG de 2e ordre qui sont équivalentes a la représentation enveloppe-phase de x. Une EDG de 4e ordre pour un signal FM double, appropriée pour la compression des phonèmes vocaliques, est aussi presentée.

ABSTRACT

A differential equation (DE) which has the signal x as its solution is a Generating DE (GDE) of x. GDEs whose coefficients vary more slowly than x can be used for data compression. Two 2nd order GDEs are presented which are equivalent to the envelope-phase representation of x. By decimating and quantizing the coefficients data compression is obtained. A 4th order GDE for a double FM signal, suitable for compression of vowels, is also shown.

1. INTRODUCTION

In our paper we deal with a new approach to data compression. We consider the signal as deterministic, or a predictive random process (in the sense of [1]), and we use such structural properties as the relationships between its derivatives, in order to obtain new signals, which are "data compressed", and at the same time sufficient for reconstructing the original signal. The key point here is that the new signals are more slowly varying, and therefore, a smaller sampling rate is required.

Let x(t), $0 < t < T$, be a three times differentiable signal. The signal can be represented with a given precision by N_x samples:

$$N_x = 2 B_x T$$

where T is the interval during x is represented and B_x is the conditioned highest frequency of x, i.e. the bandwidth of x(t). The main idea of our approach consists in representing the original signal by the three new signals $a_2(t)$, $a_1(t)$, $a_0(t)$ which constitute the varying coefficients of a second order linear homogeneous differential equation:

$$a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0 \qquad (1)$$

This equation, together with two initial conditions:

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0$$

determines x(t) as the unique solution. Therefore, (1) will be termed the Generating Differential Equation (GDE) of x(t). Let us assume that the coefficients a_k have bandwidth B_k. Then they can be represented by $N_k = 2 B_k T$ samples each. If B_k are such that $N_0 + N_1 + N_2 < N_x$, then data compression is possible. Actually, a stronger condition is required, which takes into consideration the quantization of each sample. If M_k, M_x denote the number of bits used to represent each sample of the signals a_k, x, then the condition for data compression is:

$$B_0 M_0 + B_1 M_1 + B_2 M_2 < B_x M_x \qquad (2)$$

Our aim is to find such a GDE as to satisfy (2) and thus achieve data compression. Later it will be shown that in some cases it is sufficient to use only two coefficients, and (2) should be modified accordingly.

In the past, the idea of using differential equations for data compression has been mentioned in a control theory context [2]. However, that paper did not deal with specific methods for constructing such equations for given signals. Therefore, the approach is new, and has not been treated except in some of our earlier works [3],[4].

A general comment is in order here. Traditionally, equations of the form (1) are written in the following form:

$$\ddot{x} + b_1 \dot{x} + b_0 x = 0 \qquad (3)$$

where the new coefficients have been obtained by dividing the coefficients a_k by a_2. Equations of type (1) are termed General GDE (GGDE), and of type (3) - Reduced GDE (RGDE). From the mathematical point of view, these equations are equivalent, having the same solutions. But from the point of view of data compression they are not, because their respective coefficients differ widely and <u>essentially</u> in their bandwidths. In the following, we deal with a special type of GGDE.

2. A 2ND ORDER GDE RELATED TO THE ENVELOPE-PHASE REPRESENTATION

The following GDE of order 2 was given in [4] :

$$a_2(t)\, \ddot{x} + a_1(t)\, \dot{x} + a_0(t)\, x = 0 \qquad (4)$$

$$a_2 = x\, \dot{\hat{x}} - \dot{x}\, \hat{x}$$

$$a_1 = \ddot{x}\, \hat{x} - x\, \ddot{\hat{x}} \qquad (5)$$

$$a_0 = \dot{x}\, \ddot{\hat{x}} - \dot{x}\, \ddot{\hat{x}}$$

Fig. 1

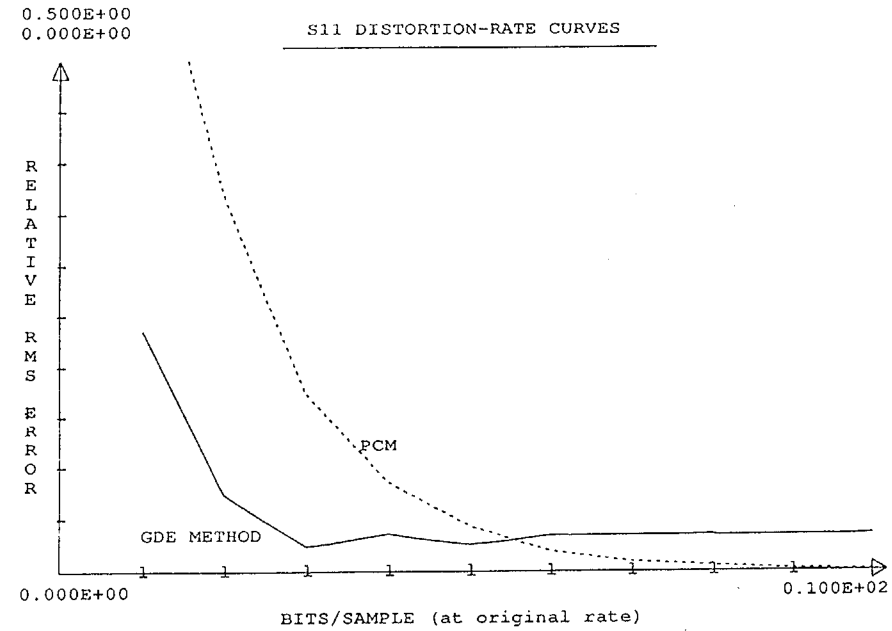
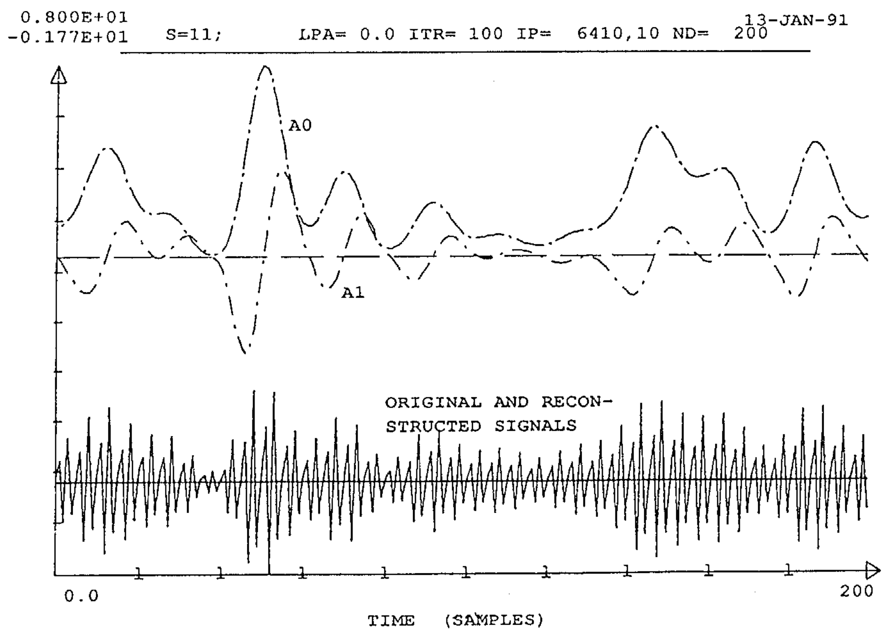


Fig. 2



where $\hat{x}$ is the Hilbert transform of signal x(t). This GDE was derived from the envelope-phase representation:

$$x = A(t) \cos \Phi(t) \tag{6}$$

It is easily seen that for this GDE the following relationship holds:

$$a_1 = - \dot{a}_2 \tag{7}$$

and therefore only the signals $a_2(t)$, $a_0(t)$ are required to retain full information about x(t). Moreover, although a similar simple relationship between a_2, a_0 cannot be found, the latter coefficients are strongly correlated. In particular, for the case when x is a multiple sinusoid, it can be shown that a_2, a_0 have components at the same frequency [4]. Therefore, greater data compression can be achieved if, instead of a_0, another signal is retained:

$$e(t) = a_2(t) - \tilde{a}_0(t) \tag{8}$$

where $\tilde{a}_0$ is a scaled version of a_0, so as to bring a_0 in the same range of values as a_2 and minimize AVERAGE($e^2(t)$):

$$\tilde{a}_0 = \alpha\, a_0 + \beta \tag{9}$$

The encoding procedure therefore includes:

a) computing a_2, a_0;

b) computing α, β, e(t);

c) decimating and quantizing a_2 (t), e(t).

The reconstruction procedure includes:

a) interpolating $a_2(t)$, e(t) to obtain signals at the original rate which is suitable for x(t);

b) computing $a_0(t)$ from $a_2(t)$, e(t), α, β;

c) computing $a_1(t)$ from a_2 according to (7);

d) solving equation (4) with initial conditions x(0), x(0).

Of course, the process of solving the GDE cannot be advanced in time indefinitely, because of the accumulating error involved in the integration process. At given intervals new values of inital conditions must be supplied to reset the error.

It has been argued that, due to the sensitivity of the GDE to errors in the coefficients, their quantization should be so fine that the large number of bits used per coefficient sample would anihilate whatever savings have been achieved by lowering the sample rate. However, simulations have shown that this is not the case. An example of a distortion-rate curve compared to PCM is shown in Fig. 1. The signal is an amplitude-and-phase modulated sinusoid, the modulating signals being two normally distributed processes of bandwidth 0.05, the center frequency 0.4, the FM β=1, the sampling frequency 1. Fig. 2 shows 200 samples of the original and reconstructed signals (undistinguishable), and two of the coefficients. The coefficients were quantized by 10 bits/samples each and decimated by a factor of 4 at the encoder, and then interpolated by 64 at the receiver end.

3. SOME LIMITATIONS OF THE ABOVE GDE AND A NEW GDE TYPE OVERCOMING THEM

We shall analyze here the amount of bandwidth reduction obtainable by the above GDE and consider ways to improve it. The coefficients in (5) can be reformulated in another way which affords a better insight in the relationship between them and the envelope and phase representation, and also allows to estimate the amount of bandwidth reduction.

LEMMA. The coefficients (5) are related to the A(t), Φ(t) by:

$$a_2 = A2\, \dot{\Phi}$$

$$a_0 = A^2 \dot{\Phi}^3 + A\,(\dot{A}\, \dot{\Phi} - \dot{A}\, \dot{\Phi}) + 2\, \dot{A}^2\, \dot{\Phi} \tag{10}$$

THEOREM. If the envelope and phase are bandlimitted in B_A, B_Φ, respectively, then the coefficients a_2, a_0 are bandlimitted in $2\,B_A + B_\Phi$.

The theorem is a consequence of the lemma and of the theorem on the bandwidth of the product of two bandlimitted signals. This does not take into account the additional data compression achieved by using e(t) instead of a_0 (t).

The above result shows that better bandwidth reduction can be achieved by simply transmitting A(t), Φ(t) and computing a_2, a_0 by (10) in the decoding stage.

However, there is another problem as well: the zero-crossing by a_2. It has been found, that in certain cases, a_2 (which is most of the time positive), crosses sometimes the zero-axis for a brief time interval. This is due, not to A(t), which is always positive, but to the other factor in (10)- Φ, and is correlated with large variation rates of the phase. The zeroing of a_2 is destructive for the process of solving the differential equation, because all existing methods assume that the leading coefficient of the equation is non-zero. It must be emphasized that the problem is a numerical-technical one, not a question of the existence of the solution: this exists and is well behaved, it being the original signal.

In order to overcome the above mentioned problem, we shall derive a new type of GDE and shall proceed as follows. Any DE solver does not solve the 2nd order equation (4) directly, but the equivalent set of two 1st order equations:

$$\dot{y} = -\frac{a_1}{a_2} y - \frac{a_0}{a_2} x \qquad \dot{x} = y \tag{11}$$

We seek an equivalent set of equations, in which the information carrying coefficients are more "symmetrically" distributed between the two equations and, hopefully, more simple. We are especially interested in removing from the denominator vanishing functions like a_2. Such a set of equations is:

$$\dot{x} = \delta_A x - \dot{\Phi} y \qquad \dot{y} = \dot{\Phi} x + \delta_A y \qquad (y = \hat{x}) \tag{12}$$

Here δ_A denotes A/A (the dissipant of A according to the terminology of [4]). Note that the denominator of δ_A does not contain the phase derivative and therefore there is no more a zero-crossing problem.

For this equation, the initial conditions are x(0), y(0).

4. A 4TH ORDER GDE FOR A DOUBLE FM SIGNAL

In this section we present a special type of 4th order GDE, corresponding to a double FM signal. The motivation behind this is as follows. In [5] it is shown that speech can be compressed by modelling the consonants as AM modulated by noise and vowels by a sum of two FM signals with constant parameters which are determined empirically by analysis. In the case of AM signal we have shown that it can be alternatively described by a GDE of type [12]. In the double FM case, the previous GDE is also valid, but is not data compressive if the two carrier frequencies are widely separated. Therefore for efficient data compression of this signal family, a 4th order GDE must be derived.

THEOREM. A GDE for the double FM signal:

$$x = a \cos \Phi(t) + b \cos \theta(t)$$

is:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} p & -q & -p & p \\ q & p & -p & -p \\ p & p & -p & -q \\ -p & p & q & -p \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \tag{13}$$

where $p = \dot{\theta} - \dot{\Phi}$, $q = \dot{\Phi} + \dot{\theta}$, $y = \hat{x}$, $w = \hat{z}$

and z is the output of a transform which discriminates between the two components of x and Hilbert-transforms only the high frequency one (cos θ).

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