# DYNAMIC BEHAVIOR IN A DIGITAL FILTER WITH SATURATION-TYPE ADDER OVERFLOW CHARACTERISTIC

**Maciej J.Ogorzałek**

**Department of Electrical Engineering**
**University of Mining and Metallurgy**
**al.Mickiewicza 30, 30-059 Krakow , Poland**

Abstract - We present a detailed two-parameter analysis of behaviour of a second-order digital filter implemented with saturation arithmetic. In our earlier studies we found that the parameter plane is devided into Arnold tongues - regions within which various types of periodic and quasiperiodic orbits exist. In this study we present some new results showing the rules for creation of fine structure in the parameter space. Apart from the Farey rule confirmed for the changes of rotation numbers of orbits born when varying one of filter parameters we found interesting rules for the regions in which the problem reduces to a map of a polygon into itself - the structure of Arnold tongues predetermines the type of polygon. Furthermore we study bifurcation phenomena taking place at the nodes of a particular type of Arnold tongue (so-called "saussage structures").

## Introduction

Throughout this paper we will consider a second order digital filter employing saturation arithmetic. The structure of such a filter realised in the direct form and the characteristic of the adder are shown in Fig.1. Equations describing its dynamics are of the form :

$$x_1(k+1) = x_2(k) \quad (1)$$

$$x_2(k+1) = F[bx_1(k) + ax_2(k) + u(k)] \quad (2)$$

Where : $a, b \in R$, $F : R \rightarrow R$ , $F(\sigma) = \sigma$ for $|\sigma| < 1$, $F(\sigma) = 1$ for $\sigma \geq 1$, $F(\sigma) = -1$ for $\sigma \leq -1$ (Note that no quantisation effects are included - this characteristic takes into account overflow only), $u(k)$ - input signal for the filter. In the present study we analyse the zero-input dynamic behavior of the filter ($u(k) = 0$).

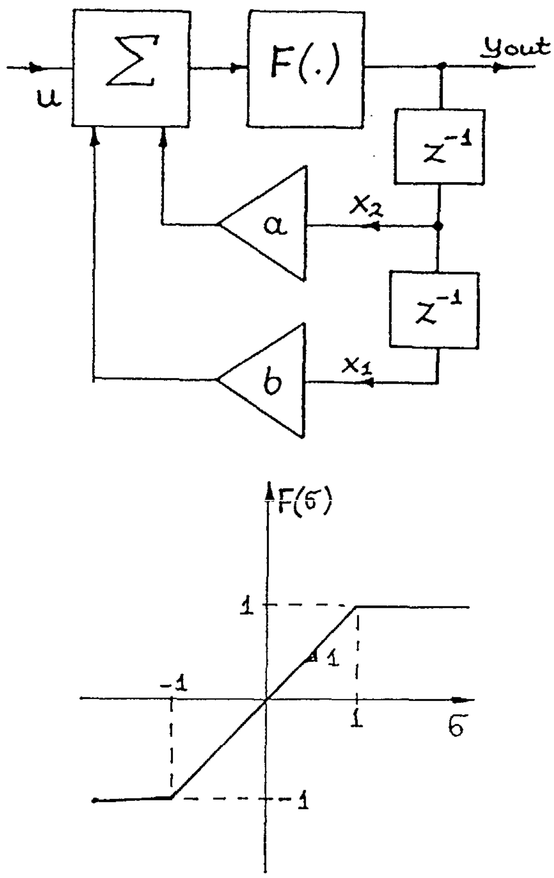


Fig. 1. Structure of the second order digital filter and overflow characteristic of the accumulator.

## Structure of the bifurcation plane
## Arnold tongues

Dynamic behaviours of this class of systems have been studied by a number of authors. Basic stability analysis can be found in [5]. Willson [9] has shown that the nonlinear system under consideration is asymptotically stable for all parameters $a$ and $b$ within the linear stability triangle ($F(x) = x$) ie. $b > -1$, $b < a + 1$ and $b < -a + 1$ (this triangle is indicated in Fig.2. In conclusions of their paper on chaos in digital filters Chua and Lin [2] left several open problems concerning dynamic behaviors in the filter with saturation arithmetic. Some of these problems were addressed in our earlier studies [3]-[8]. We found that depending on the actual filter coefficient values a variety of oscillatory orbits could exist. Fig. 2 shows regions of existence of principal types of such orbits with different rotation numbers [4] on the $a - b$ parameter plane. Outside the linear stability triangle one can clearly see the Arnold tongue structure - the pricipal "tongues" 1/2 and 0/1 being the half- planes $b > a+1$ and $b > -a+1$ respectively. The quadrant delimited by $b < a-1$ and $b < -a-1$ constitutes the 1/4 tongue. It is possible to show that the borders of 1/3 tongue asymptotically approach the borders of 1/2 and 1/4 tongue. In a similar way the curves delimiting the 1/6 tongue approach asymptotically the lines delimiting the 1/4 and 0/1 region. It is easy to notice the existence of an infinite number of (shrinking by 1/2) half- circular regions (comp Fig.2) in which the convergence towards periodic orbits is very slow and these orbits could be detected only after a large number of iterates (> 25000).

Figure 3 shows in more detail the structure of the largest half- circular region found in the earlier experiment. This time we continued the experiment for 50000 iterates for each chosen $a$ and $b$. Further extension of the observation interval above 50000 iterates reveals even finer structures of the Arnold's tongues of so- called "saussage structure" and aparently infinite number of curves defined by nodes of the tongues (points where the successive "saussages" representing solutions of the same rotation number touch each- other). The "saussage" pattern has a self-similar structure - repetes in every half- circular region. Below specified accuracy level $\varepsilon$ for filter parameters it is not possible to specify what will be the observed system behaviour. The filter exhibits "final state sensitivity" to parameter changes.

We noticed also in [8] that there are several regions in the parameter space where more then one stable orbits coexist. A more detailed conclusion follows from recent analysis. There are always:
-). single stable-unstable pair of orbits of (the same) even period,
-). two pairs of stable-unstable orbits of (equal) odd pariod.

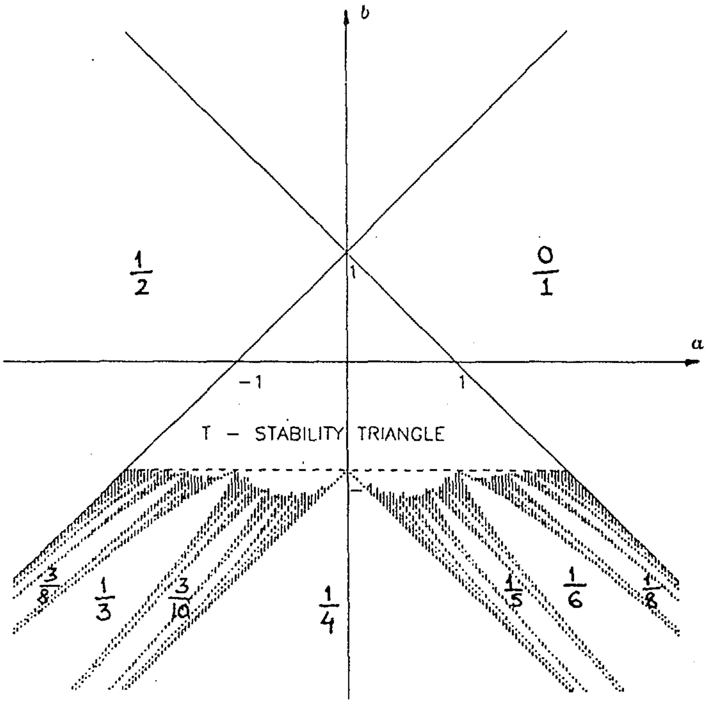


Fig. 2. Two-parameter bifurcation diagram for the digital filter employing saturation arithmetic. Asymptotic stability triangle and principal Arnold tongues are shown.

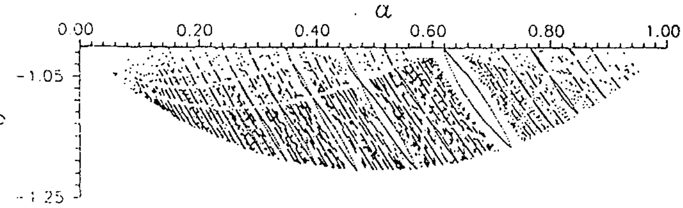


Fig. 3. Detail of the largest half- circular region in the parameter space revealing finer structure of the Arnold tongues - "saussage structures".

## Devil's staircase

We studied in detail how the rotation number of system orbits changes when varying $a$ for fixed $b$ value. These diagrams reveal typically the devil's staircase structure as shown for example in Fig.4. Rotation number is a monotonic continuous function with plateaus of finite width at every rational value [1]. Zooming-in this picture shows repeating finer structures of this kind (self-similarity). Changes of rotation numbers obey the Farey's rule - between two cycles of rotation numbers $\frac{p}{q}$ and $\frac{r}{s}$ there is always an orbit with rotation number $\frac{p+r}{q+s}$.

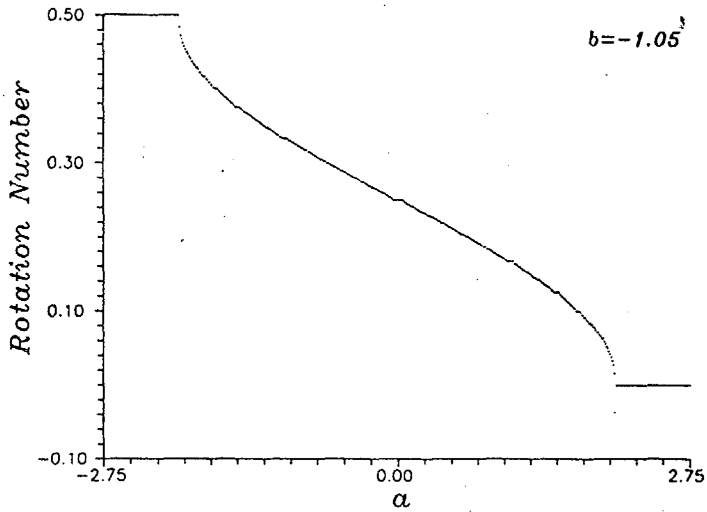


Fig. 4. Change of rotation number as a function of the parameter $a$ (for fixed $b$) revealing typical devil's staircase structure.

## Relation between properties of the one-dimensional map and the Arnold tongues structure

In our recent paper [7] we proved several theorems concerning invariant sets and limit sets of system trajectories in various parameter ranges. In particular we have shown that for the parameters in the set :
$\Omega = \{(a,b) \in R : b < -1, |b+a| < 1 \text{ or } |b-a| < 1\}$

(see Fig. 2) there exists a one- dimensional invariant set attracting all system trajectories and the study of dynamics of our system can be reduced to analysis of a map of a polygon into itself (homeomorphic with a map of a circle). In the limit case number of sides goes to infinity. This one- dimensional map is non- decreasing which implies existence and uniqueness of a rotation number for any parameters $a$ and $b$ chosen from the specified range (see [4]). Thus we excluded the possibility of overlaping of the Arnold tongues within the sets $\Omega$ and excludes possibility of existence of chaotic motion within this range of parameters. However that it is possible to tune the filter parameters in such a way that oscillations of any chosen period can be generated. Studying the complexity of the invariant sets we were able to find in the parameter space regions of existence particular types of the polygons with prescribed number of sides. Comparison of the Arnold tongues diagram and the regions of existence of invariant polygons diagram (Fig.5) suggests that in the region below the self-similar circles the number of sides of invariant polygons

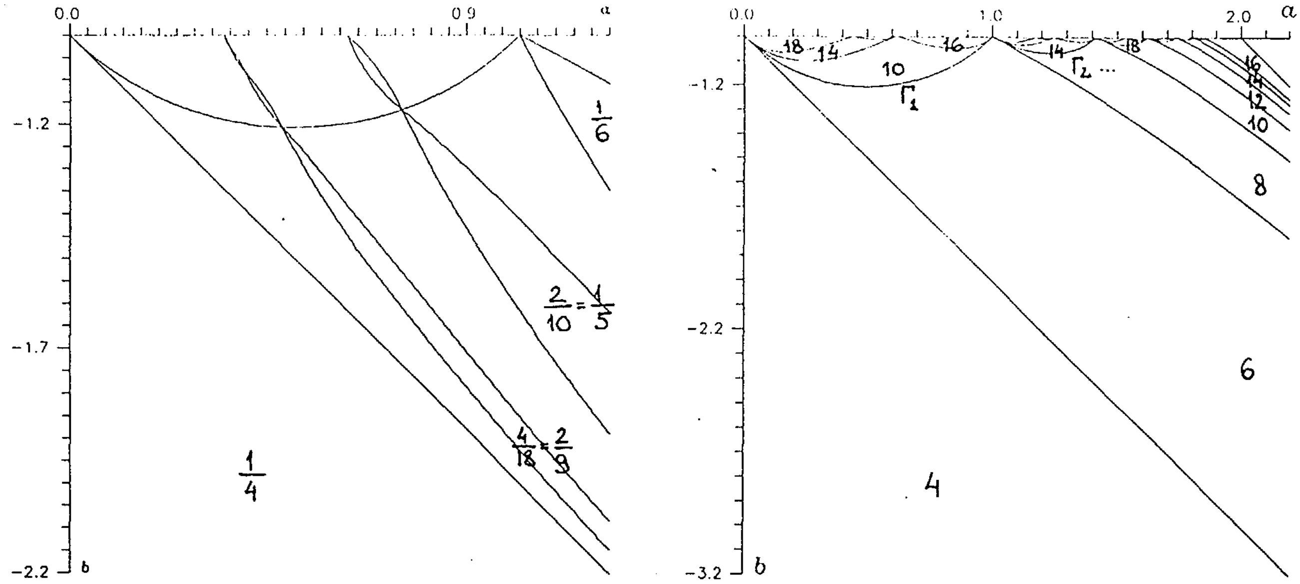


Fig. 5. Comparison of a magnified fragment of the Arnold tongues diagram and the regions of existence of invariant polygons diagram. Orbits of the same rotation number exist in distinct invariant sets.

just change by 2 in the regions touching each-other. However depending on the choice of parameters the limit sets are varied within a chosen polygon type. Furthermore it is possible to find the rule for the number of sides of the invariant polygon in a considered region within the "circular" range.

Note that the $i-th$ curve $\Gamma_i$ delimiting chosen $i-th$ "circular") region has always both ends on the line $b=-1$ and at each of these points and Arnold tongue has its origin - thus at the ends of any delimiting curves two distinct Arnold tongues originate eg. the $\frac{p}{q}$ and $\frac{r}{s}$ tongues . Then in the region delimited by $\Gamma_i$ there exists an invariant polygon with $q+s$ sides. Note however that depending on the actual coefficient values there are limit sets of the same kind (period) in invariant sets of distinct types eg. stable period-nine orbit on a hexagon, decagon etc. Numerical experiments have indicated that despite the same rotation number orbits existing in different parts of chosen "saussage" structure are different. In the following section we will look into some detail of bifurcation phenomena encountered at the nodes of "saussages".

## Bifurcation phenomena at the nodes of Arnold tongues

As a typical example let us consider a $\frac{2}{9}$ Arnold tongue shown in detail in Fig.6. It has the "saussage" structure composed of four sections. Each section belongs to the region of the $a-b$ plane with different invariant polygon (6, 10, 14 etc. sides) - compare Fig.5. For the purpose of study of bifurcation phenomena at the nodes of this Arnold tongue we constructed a specific bifurcation diagram changing both $a$ and $b$ in along lines connecting the nodes. The resulting diagram is shown in Fig.7.

In Fig.7 both stable and unstable orbits are indicated. Looking into details reveals that stable orbits existing within chosen subregion destabilise when crossing to the next one through the node of the "saussage" and at the same time new stable orbits of the same rotation number are born. We could call such phenomenon stability exchange. To show that the orbits in each section of the "saussage" are different we found the symbolic characterisation of each of them. Defining the symbol sequences as proposed in [2] :

$$s_k = \begin{cases} -1, & \text{if } bx_1(k) + ax_2(k) \geq 1 \\ 1, & \text{if } bx_1(k) + ax_2(k) < -1 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

stable orbits existing in each section (starting from the line $b=$

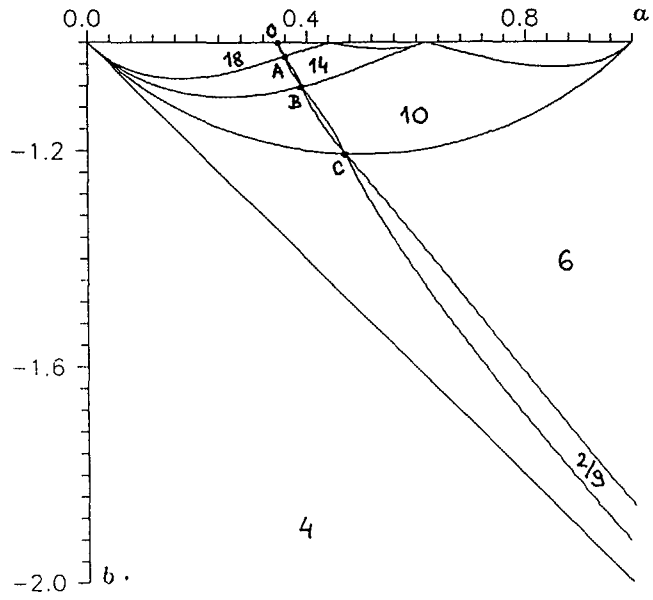


Fig. 6. The $\frac{2}{9}$ Arnold tongue - "saussage" structure composed of four sections. The curves delimit regions of existence of distinct types of invariant polygons.

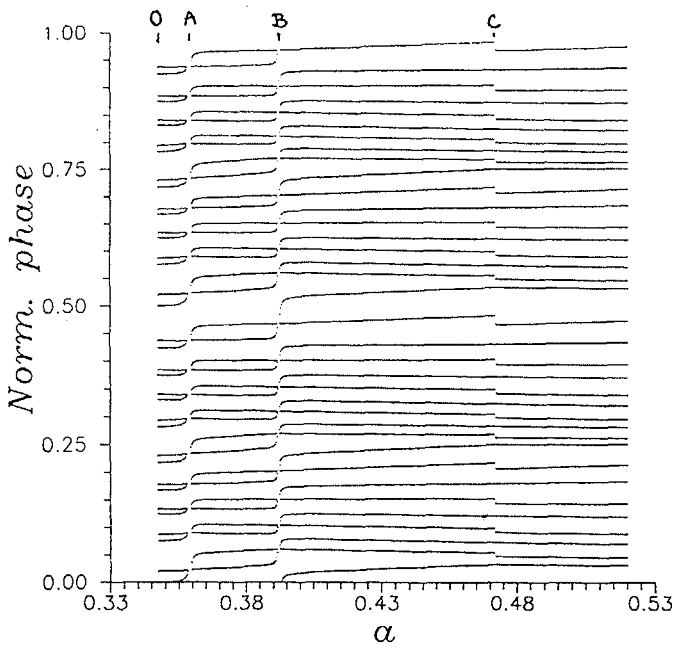


Fig. 7. Evolution of the period-9 orbits when changing parameters $a$ and $b$ within the $\frac{2}{9}$ Arnold tongue. Both stable and unstable orbits are indicated. Note the "jumps" when crossing the nodes of the saussage structure.

$-1$) of the $\frac{2}{9}$ Arnold tongue are characterised by the following symbol sequences:
$(1,0,0,0,0,0,0,0,0)$ and $(-1,0,0,0,0,0,0,0,0)$ (these orbits exist at the origin of the tongue for $a = 2cos\frac{4\pi}{9} = 0.3473$, $b = -1$)
$(1,0,-1,0,0,0,0,0,0)$ and $(-1,0,1,0,0,0,0,0,0)$
$(1,0,-1,0,1,0,0,0,0)$ and $(-1,0,1,0,-1,0,0,0,0)$
$(1,0,-1,0,1,0,-1,0,0)$ and $(-1,0,1,0,-1,0,1,0,0)$
$(1,0,-1,0,1,0,-1,-1,0)$ and $(-1,0,1,0,-1,0,1,1,0)$
At each bifurcation point one non-zero symbol is added in the sequence the length of the sequence (9) remains unchanged.

## Conclusions

Our studies revealed some interesting dynamic behaviors in the filter associated with the nonlinear characteristic of the adder. The structure of the bifurcation parameter plane is composed of self- similar regions which we called "saussage structures". Furthermore the regions of existence of periodic orbits characterised by different rotation numbers form a fine Arnold tongue structure. It is possible to show that the tongues do not overlap and the filter can not exhibit chaotic trajectories. Within the tongues characterised by rational rotation numbers with even denominator there exist always single periodic orbits while in the case of odd denominator stable periodic orbits always come in pairs. One-parameter bifurcation diagrams reveal typical devil's staircase route of changes of rotation numbers when varying the bifurcation parameter. There is an interesting relation between properties of the one dimensional map describing the system's behavior and structure of the bifurcation parameter plane. The complexity of the polygon maped is related to denominators of rotation numbers in the tongues associated with considered region.

Bifurcation phenomena within the Arnold tongues encountered when passing from one "saussage" to the following one through the nodes are of the stability exchange type. Symbol sequences characterising the orbits differ by one non-zero symbol when passing between "saussages".

## Acknowledgement

This research has been supported by research grants 10.382.13 and 11.382.06 from the University of Mining and Metallurgy. This paper has been prepared in part during the author's stay at the Swiss Federal Institute of Technology, Lausanne.

## References

[1] P.Bak "The devil's staircase". Physics Today, vol.39, pp.38-45, 1986.

[2] L.O.Chua, T.Lin "Chaos in Digital Filters". IEEE Trans. CAS, CAS-35, pp.648-658, 1988.

[3] Z.Galias, M.J.Ogorzałek "Bifurcation Phenomena in Second Order Digital Filter with Saturation-Type Adder Overflow Characteristic". IEEE Trans. CAS, CAS-37, No.8, pp.1068-1070, 1990.

[4] R.S.MacKay, C.Tresser "Transition to Topological Chaos for Circle Maps". Physica 19D, pp.206-237, 1986.

[5] D.Mitra "Large Amplitude Self-sustained Oscillations in Difference Equations which Describe Digital Filter Sections Using Saturation Arithmetic". IEEE Trans. ASSP, ASSP-25, pp.134-143, 1977.

[6] M.J.Ogorzałek, Z.Galias "Arnold tongues and devil's staircase in a digital filter with saturation arithmetic". Proc.IEEE ISCAS, Singapore 1991, vol.1, pp.384-387.

[7] M.J.Ogorzałek, Z.Galias "Limit sets of trajectories in a nonlinear digital system". Presented at the Int. Symposium on Mathematical Theory of Networks and Systems, Kobe, June 17-21, 1991.

[8] M.J.Ogorzałek, E.Mosekilde "Multiple periodic orbits and structure of basin boundaries in a digital filter employing saturation arithmetic". Proceedings of the European Conference on Circuit Theory and Design, Copenhagen 1991, (in press).

[9] A.N.Wilson "Limit Cycles Due to Adder Overflow in Digital Filters". IEEE Trans. CT, CT-19, No.4, pp.342-345, 1972.