Digital machines iterating chaotic maps: Roundoff induced periodicity

Christian Beck
Institute for Theoretical Physics, Technical University of Aachen, W-5100 Aachen, Germany

Abstract

Due to roundoff the orbits of maps on a finite phase space necessarily become periodic when a digital computer is used for iteration. In typical cases the average period length scales with the machine precision, the roundoff scaling exponent ϵ being related to the Renyi dimensions $D(q)$ of the corresponding invariant probability measure. Under a random map assumption it can be shown that $\epsilon = \frac{1}{2}D(2)$, whereas for critical systems such as the Feigenbaum attractor $\epsilon = D(\infty)$. For certain chaotic maps the random map assumption breaks down, leading to anomalous roundoff scaling behaviour.

1. Introduction

The rapid development in the field of nonlinear dynamics during the past fifteen years was only possible due to the use of computers. In fact, most papers in nonlinear science are inspired by numerical experiments, only in rare cases it is possible to analyse a nonlinear dynamical system without using a computer. Hence it is reasonable to ask for the cumulative effects of roundoff errors, especially with respect to the sensitive dependence on initial conditions for chaotic systems. Many authors have already dealt with this question ([1]-[10], and references therein).

Computer roundoff can be regarded as a discretization of the phase space. We are familiar with the enormous effects that can be produced by a discretization of *time*: For example, the continuous-time dynamical system

$$\dot{x} = \mu x(1-x) \qquad (1)$$

has the simple solution

$$x(t) = \frac{1}{1+Ce^{-\mu t}} \qquad (2)$$

i.e., $x(t)$ asymtotically approaches a fixed point for arbitrary μ. On the other hand, the discrete-time dynamical system

$$x_{n+1} = \mu x_n(1-x_n) \qquad (3)$$

exhibits, depending on μ, bifurcations, chaos, and many other interesting nonlinear effects. Can a spatial discretization cause similar drastic effects as a temporal discretization does? The answer is yes. As a very simple example let us consider the binary shift map

$$x_{n+1} = 2x_n - \lfloor 2x_n \rfloor \qquad (4)$$

$x \in [0,1]$, $\lfloor\ \rfloor$: integer part. This system is known to exhibit strong chaotic behaviour: It is ergodic, even mixing. However, if the above recurrence relation is naively implemented on a computer, one notices that after a few iterations all initial values fall onto the fixed point $x^* = 0$. For the exact system this fixed point is unstable, i.e. it does not attract trajectories. However, for the truncated system the behaviour is totally different: Due to the finite precision of the computer only a finite number of digits of the initial value is stored, and as the map (4) acts as a shift, after a few iterations all digits are "shifted away", the orbit falling onto the fixed point $x^* = 0$. This example is somewhat extreme, the problems are caused by the special piecewise linear properties of the binary shift map. In general, however, we should be aware of the fact that roundoff errors can completely change the behaviour of a nonlinear map.

2. Roundoff plots

It is clear that due to the finite number of phase space cells available, the orbits of any map must become periodic on a finite state machine. It turns out that in most cases the average length of the asymptotic periodic orbit is much smaller than the number of phase space cells. As the investigation of a given dynamical system with a fixed precision may lead to totally wrong conclusions, a good idea is to systematically study roundoff effects by varying the precision Δ in an artificial way and to see what is going to happen. As a model, instead of f

we may iterate the map

$$\hat{f}(x) = \Delta \lfloor f(x)/\Delta \rfloor \qquad (5)$$

For each (artificial) precision Δ we may choose a random initial point and determine the length of the periodic orbit that is approached starting with this initial value. We plot the orbit length as a function of the precision in a double logarithmic plot and repeat the experiment for a large number of different precisions and initial values. This can be done very easily for arbitrary maps f. The resulting "roundoff-plots", first introduced in [3], in a certain sense represent an ensemble of computers doing their roundoff errors in a slightly different way. The plots allow to estimate the typical discretization behaviour of a given map f.

Figs. 1-4 show these roundoff plots for various examples of maps f. In Fig. 1 we have chosen the tent map

$$f(x) = 1 - 2|x| \qquad x \in [-1, 1] \qquad (6)$$

Fig. 2 and 3 show roundoff plots of the logistic map

$$f(x) = 1 - \mu x^2 \qquad (7)$$

for the fully developed chaotic case $\mu = 2$ (Fig. 2) and the accumulation point of period doubling $\mu = 1.4011552$ (Fig. 3). Finally, Fig. 4 shows the roundoff plot of the circle map [16]

$$\Theta_{n+1} = \Theta_n + \Omega - (k/2\pi)\sin(2\pi\Theta_n) \qquad (8)$$

(mod 1) with $\Omega = (\sqrt{5} - 1)/2$ and $k = 1$. Whereas for chaotic maps such as those of Fig. 1 and 2 one typically

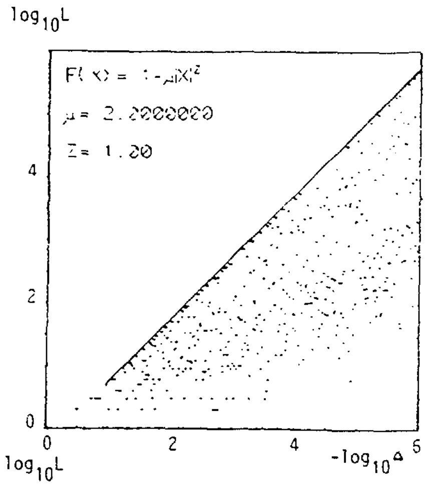

Fig. 1

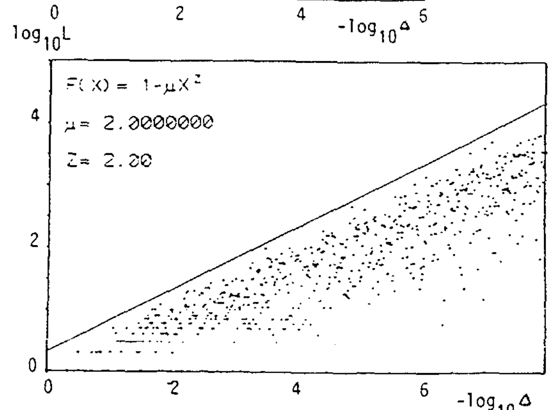

Fig. 2

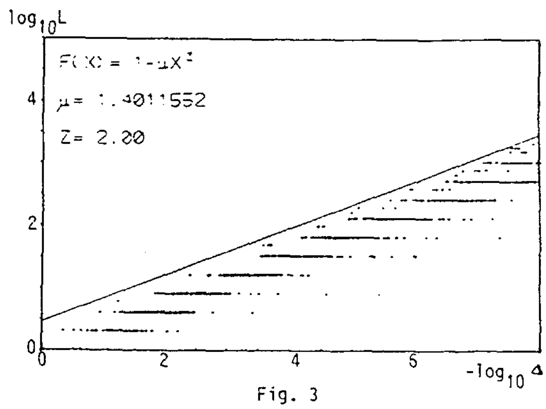

Fig. 3

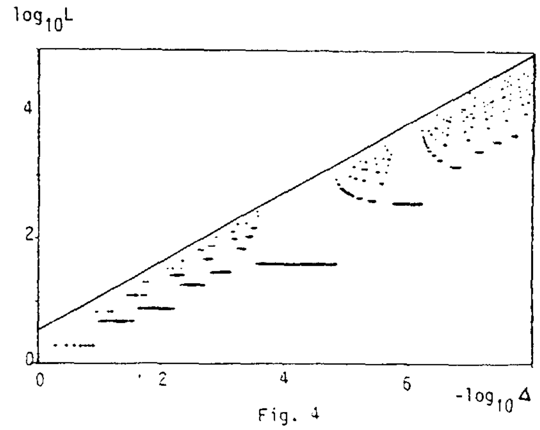

Fig. 4

observes randomly scattered period lengths, the plots for non-chaotic maps such as those of Fig. 3 and 4 typically possess some complicated structured pattern of points, the details of which still far from being fully understood. Interesting is the occurence of roundoff-induced period doublings in Fig. 3. Common to all figures is the fact that obviously the average period length $< L >$ (or, alternatively, the maximum period length) scales with the precision Δ of the machine. The corresponding roundoff scaling exponent ϵ is given by the asymptotic slope in the roundoff plots:

$$< L > \sim \Delta^{-\epsilon} \qquad (\Delta \to 0) \qquad (9)$$

3. The random map model

We are still far away from a general theory of the roundoff exponent ϵ. However, under certain reasonable assumptions of statistical independence a theoretical prediction on ϵ can be given. This is the assumtion that a discretized chaotic dynamical system can be properly modelled by a discrete random map. Such a random map is simply an ensemble of $N \times N$ matrices A with entries 0 and 1 only. A 1 at position (i, j) means that cell j is mapped onto cell i. N is the number of phase space cells and related to the precision of the computer and the phase space dimension d by

$$N \sim \Delta^{-d} \qquad (10)$$

The random map corresponds to an ensemble of computers doing their roundoff in a slightly different way. Once a member of the ensemble is chosen, its entries are kept constant during the iteration, which is simply matrix multiplication. The random map ensemble is defined by the following properties [8]:

- Each column of A (a member of the ensemble) contains exactly one entry 1, all the other entries in this column being 0.
- The probability p_i to find the entry 1 in row i is the same for each *column* j.
- The probabilties p_i may be different for different *rows* i.
- The position of the 1 in column k and column j are independent events for $k \neq j$.

Property 1 simply reflects the fact that a computer uniquely maps a phase space cell j onto another phase space cell i when a map is iterated. Property 2 is expected to be true for strongly mixing maps after a short number of iterations: The trajectory has forgotten its initial value, represented by the column index j. Property 3 states that the invariant measure of a chaotic map usually is not the uniform distribution. We assume that even if there are small roundoff pertubations, the probability to find the iterate in a certain subregion of the phase space is still given by the (natural) invariant measure of the map [5,6]. Property 4 reflects the sensitive dependence on initial conditions. After a short time of iterations the actual position of one trajectory is independent of that of another trajectory corresponding to another initial value.

No doubt, the above assumptions are just model assumptions for a discretized chaotic map after a short number of iterations. The crucial point for a random map to be a good model is that the mixing property (i.e., the asymptotic independence of events) of the exact map is not destroyed by the discretization process. Nevertheless, given the random map assumptions one can do rigorous mathematics and proof the following [8]:

1. The probability $q(L)$ that an orbit terminates in a periodic orbit of length L is given by

$$q(L) = \sum_{k=L}^{N} \sum_{i_1,i_2,\ldots,i_k}^{*} p_{i_1}^2 p_{i_2} \cdots p_{i_k} \quad (11)$$

The star indicates that all indices $i_1, \ldots, i_k$ are different.

2. For $\Delta \to 0$ the average length $< L >= \sum_{k=1}^{N} Lq(L)$ scales as

$$< L > \sim \Delta^{-\frac{1}{2}D(2)} \sim N^{\frac{D(2)}{2d}} \quad (12)$$

Here

$$D(2) = \lim_{\delta \to 0} \frac{1}{\log \delta} \log \sum_i p_i^2 \quad (13)$$

is the correlation dimension of the measure μ (δ: cell size):

$$p_i = \int_{i-th\ cell} d\mu(x) \quad (14)$$

3. The probability distribution of the transient length T (i.e., the orbit length until the asymptotic periodic cycle is reached) coincides with the probability distribution of the period length L. In particular, the averages $< T >$ and $< L >$ coincide.

Eq. (12) was first conjectured in [7], a rigorous proof was given in [8]. Numerical experiments indicate that for generic chaotic maps the above consequences of the random map assumption are approximately satisfied. Hence the random map model seems to be quite an appropriate tool to deal with chaotic discretized maps. Exceptions are piecewise linear maps, for which the above assumptions of statistical independence can be violated. We will deal with this in section 5. It should be clear that there are many further interesting questions that can be asked with respect to discrete dynamics. For example, an interesting problem is the average number of coexisting cycles of the discretized map. At least for random maps with uniform distribution $p_i = 1/N$ it is known [1,4] that this number is quite small and grows logarithmically with N. In practice one observes that there is just one or at best a few cycles that attract almost all initial values, whereas the remaining cycles appear to be less important.

4. Critical systems

It is quite clear that random maps can only be a good model for discretized dynamical systems with positive Lyapunov exponents. For critical systems, however, such as the logistic map at the accumulation point of period doubling, the Lyapunov exponent vanishes and we have to look for a different model to describe the discretization behaviour, as the random map assumptions are not valid any more. It turns out that for these systems still the Renyi dimensions $D(q)$ defined by

$$D(q) = \lim_{\delta \to 0} \frac{1}{q-1} \frac{\log \sum_i p_i^q}{\log \delta} \qquad (15)$$

determine the roundoff scaling behaviour, this time, however, the limit dimension $D(\infty)$ is relevant. For a system with Lyapunov exponent 0 little pertubations such as roundoff errors will not grow exponentially fast during the iteration process. Rather, we expect that the difference between the exact orbit and the rounded orbit will be of the order of the precision Δ of the machine. This means that the maximum orbit length is determined by the minimum distance d_{min} between orbit elements of the exact map: As soon as $\Delta > d_{min}$, the orbit cannot be resolved. For the Feigenbaum attractor of a single humped map with a maximum of order z, let $d_{min}^{(n)}$ denote the minimum distance of 2^n subsequent orbit elements on the attractor. This distance scales as

$$d_{min}^{(n)} \sim \left(\frac{1}{\alpha^z}\right)^n \qquad (16)$$

where $\alpha = \alpha(z)$ is the Feigenbaum constant [11]. From $\Delta \sim d_{min}^{(n)}$ and $< L > \sim 2^n$ we obtain

$$< L > \sim \Delta^{-\epsilon} \qquad \epsilon = \frac{1}{z \log_2 \alpha} \qquad (17)$$

Expressed in terms of the Renyi dimensions, $1/(z \log_2 \alpha)$ is just the Renyi dimension $D(\infty)$ of the Feigenbaum attractor. In general, for arbitrary dynamical systems $D(\infty)$ describes the scaling behaviour of the region of the phase space where the invariant measure is most concentrated. In this region the discretized orbit of a critical system becomes periodic, as here the distance between orbit elements takes its minimum value. Thus, for these types of systems we expect that in general the roundoff scaling exponent is given by

$$\epsilon = D(\infty) \qquad (18)$$

In [3] we have tested eq. (18) for the Feigenbaum attractor for various values of z and found very good coincidence between the values of $D(\infty)$ and the slope in the roundoff plots. Moreover, it turns out that for *arbitrary* $z \geq 1$ there is a "super-universal" constant ϵ_0 that bounds the exponent $D(\infty)$ from above [3]:

$$D(\infty) \leq \epsilon_0 = 0.38165305 \qquad (19)$$

$D(\infty)$ takes its maximum value ϵ_0 for $z = 1.6922$. The "physical" meaning of the constant ϵ_0 is the following: No matter which universality class is chosen, there is a principle bound on resolving the Feigenbaum attractor. With a digital machine of precision Δ it is only possible to see orbit lengths that satisfy $L < const\ \Delta^{-\epsilon_0}$. Notice that this is less than square root behaviour.

5. Breakdown of the random map assumption

It is quite clear that critical discretized systems do not satisfy the random map assumptions, because the exact system is non-chaotic. This results in the fact that the roundoff scaling exponent is given by $D(\infty)$ rather than $\frac{1}{2}D(2)$. What is more surprising is the fact that even if the exact dynamical system has strong mixing properties, these can be destroyed by the discretization process, leading to a breakdown of the random map assumption and to anomalous roundoff scaling. As an example, let us consider the class of maps on the interval $[-1,1]$ defined by

$$x_{n+1} = 1 - 2|x_n|^z \tag{20}$$

where $z \in [1,\infty)$. A general theorem of Misiurewicz [12] guaranties the existence of an absolute continuous invariant ergodic measure. The Renyi dimensions are given by [13,14,15]

$$D(q) = \begin{cases} 1 & q < q_c \\ \frac{1}{z}\frac{q}{q-1} & q \geq q_c \end{cases} \tag{21}$$

where $q_c = z/(z-1)$. In particular, we obtain

$$\frac{1}{2}D(2) = \begin{cases} \frac{1}{2} & z < 2 \\ \frac{1}{z} & z \geq 2 \end{cases} \tag{22}$$

and

$$D(\infty) = \frac{1}{z} \tag{23}$$

Numerically one indeed observes in a roundoff plot the slope $\epsilon = \frac{1}{2}D(2)$ for $z > 1$, at least within statistical and systematic errors. However, when z approaches 1, the behaviour of the discretized system suddenly changes: Suddenly there are very long periods, of the same order of magnitude as the number of phase space cells available (see Fig. 1). Indeed, the numerical results indicate that the slope in the roundoff plots is given by[1]

$$\epsilon = \begin{cases} 1 & z = 1 \\ \frac{1}{2} & 1 < z < 2 \\ \frac{1}{z} & z \geq 2 \end{cases} \tag{24}$$

The long period lengths occuring for $z = 1$ can be explained by the fact that one possible discretization scheme is to map each cell of the phase space onto a different phase space cell, i.e. no preimages coincide (this is possible due to the piecewise linear properties of the map). Then the length of the periodic orbit must be of the same order of magnitude as the number of phase space cells available, which leads to $\epsilon = 1$. Actually, the situation for $z = 1$ is somewhat more complicated: The large orbit lengths are only observed if we use the artificial discretization scheme eq. (5). However, if we directly iterate the tent map on an IBM machine, all orbits again fall onto the fixed point $x^* = -1$, for the same reason as they do for the binary shift map (see section 1). From this point of view, we can also define $\epsilon = 0$ for $z = 1$. In any case, we notice that at $z = 1$, due to the piecewise linear properties of the map, the random map assumption is not valid anymore. This results in the fact that the roundoff exponent depends on details of the discretization scheme and can take values different from $\frac{1}{2}D(2)$. This complicated behaviour, as well as the possible breakdown of the random map assumption, has been overlooked in [7]. We notice that there are two "critical" points, where the roundoff scaling exponent exhibits kind of a phase transition behaviour: At $z = 2$ it is not differentiable with respect to z. This is just a consequence of the known phase transition behaviour of the Renyi dimension $D(2)$ [13,14,15]. Moreover, there is another critical point $z = 1$ with discontinuous behaviour, which is a consequence of the breakdown of the random map assumption.

[1] In [3] we only investigated integer z, for which $\epsilon = \frac{1}{z}$

References

[1] Y.E. Levy, Phys. Lett. **88A**,1 (1982)

[2] J.L. McCauley Jr., J.I. Palmore, Phys. Lett. **115A**, 433 (1986)

[3] C. Beck, G. Roepstorff, Physica **25D**, 173 (1987)

[4] B. Derrida, H. Flyvberg, J. Phys. **48**, 971 (1987)

[5] N. Friedman, A. Boyarsky, M. Scarowsky, Comput. Math. Applic. **15**, 997 (1988)

[6] P. Góra, A. Boyarsky, Comput. Math. Applic. **16**, 321 (1988)

[7] J.A. Yorke, C. Grebogi, E. Ott, Phys. Rev. **38A**, 3688 (1988)

[8] C. Beck, Phys. Lett. **136A**, 121 (1989)

[9] P. Grassberger, in 20th IFF Ferienkurs "Computersimulationen in der Physik", KFA Jülich (1989)

[10] D.J.D. Earn, S. Tremaine, Preprint (1991)

[11] M.J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978)

[12] P, Collet, J.P. Eckmann, Iterated Maps on the Interval as Dynamical Systems, Birkhäuser (1980)

[13] E. Ott, W. Withers, J.A. Yorke, J. Stat. Phys. **36**, 697 (1984)

[14] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, B.I. Shraiman, Phys. Rev. **33A**, 1141 (1986)

[15] C. Beck, Physica **41D**, 67 (1990)

[16] S.J. Shenker, Physica **5D**, 405 (1982)