PERFORMANCE EVALUATION OF ESTIMATES OF HIGH ORDER MOMENTS: APPLICATION TO TIME DELAY ESTIMATION IN COLOURED NOISE

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RÉSUMÉ

Dans cet article nous adressons le problème de l'évaluation analytique des performances de l'estime d'un paramètre inconnu avec les statistiques d'ordre supérieur (SOS). On a evalué d'abord la matrice de covariance des cumulants estimés sur un echantillon, qu'on a obtenue avec un algorithme pour un langage de manipulation symbolique.
Enfin on présente de résultats de l'analyse des performances de l'éstimateur du temps de retour employant les cumulants du quatrième ordre.

ABSTRACT

In this paper we address the problem of analytically evaluating the performance of parameters estimates based on Higher Order Statistics (HOS). In particular the covariance matrix of HOS sample cumulants is computed through an algorithm for symbolic manipulation packages.
Finally, the method is applied to the performance analysis of time delay estimators based on fourth-order cumulants.

1. INTRODUCTION

Application of higher order statistics (HOS) to estimation problems is widely diffused in many field of interest. While the introduction of HOS is justified by several motivations, ranging from their blind Gaussian rejection capabilities, sensitivity to the phase of linear transformations, higher detection capabilities, *etc.*, the lack of general means for performance evaluation does not allow for objective measurement of the overall *quality* of the parameter estimates and, moreover, does not allows for systematic comparison of different estimation procedures applied to the same problem. In addition, several *near-optimal* techniques base the optimization phase on the knowledge of the covariances of the estimated HOS. (see for instance [1, 2]).

The main aim of this contribution is to provide an effective means of evaluation of the covariances of cumulant's estimates obtained from a finite sample of observations. The main result consists in explicit formulas for the covariances of the sample cumulants in terms of the *true* cumulants (and moments). Strictly speaking the formulas are asymptotically exact, nevertheless they show a sufficient degree of accuracy for finite sample sizes of practical interest.

The formulas have been symbolically implemented using a standard symbolic manipulation packages (i.e. *Mathematica*), in order to provide effective computation capabilities. During the preparation of this work, an analogous tool has been employed in [3].

We remark that the obtained result is asymptotically exact; for finite sample size, the only approximation introducted is the linearization of the relation between the cumulants and the related moments. We notice that the metodology adopted here is quite general and produces a useful closed form expression.

2. HOS ESTIMATORS PERFORMANCE

Let us summarize the guidelines of the overall derivation of the above cited formulas.

The covariance matrix of sample cumulants is evaluated by relating it to the covariance matrix of sample moments.

Let N be the number of available measurements. It is

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well known that the sample moments are strongly consistent estimates of moments; the sequence of random variables obtained for increasing N is asymptotically normally distributed (see for instance [4]).

Under these assumptions, also the sequence of the sample cumulants is asymptotically normally distributed, and the covariance matrix $\mathbf{\Gamma}$ of the sample cumulants is asymptotically given by the covariance matrix of the sample moments $\mathbf{M}$, transformed, through a similarity transformation, by the Jacobian $\mathbf{J}$ of the non linear function relating the generic multivariate cumulant to a set of multivariate moments, *i.e.* $\mathbf{\Gamma} = \mathbf{J} \cdot \mathbf{M} \cdot \mathbf{J}^T$ (see for instance [4]). Thus, for large N, the computation of $\mathbf{\Gamma}$ reduces to the evaluation of $\mathbf{J}$ and $\mathbf{M}$.

In this respect, to obtain $\mathbf{J}$, we first derive the expression of the single multivariate cumulant as a function of the related moments.

Since, for a generic r.v. x, the univariate cumulant [1] in terms of the corresponding univariate moments is (see [5]):

$$\kappa_x^{(r)} = (-1)^{r-1} \begin{vmatrix} \mu_x^{(1)} & 1 & 0 & \cdot & 0 \\ \mu_x^{(2)} & \binom{1}{0}\mu_x^{(1)} & 1 & \cdot & 0 \\ \mu_x^{(3)} & \binom{2}{0}\mu_x^{(2)} & \vdots & \cdot & 0 \\ \vdots & \vdots & \vdots & \cdot & \vdots \\ \mu_x^{(r)} & \binom{r-1}{0}\mu_x^{(r-1)} & \cdot & \cdot & \binom{r-1}{r-2}\mu_x^{(1)} \end{vmatrix},$$

the multivariate cumulant can be computed by recursive application of a symbolic differentiation rule to the previous relation, as indicated in [5]. Then, the Jacobian $\mathbf{J}$ is obtained by differentiating the multivariate cumulant w.r.t. the multivariate moments.

To evaluate the covariance matrix of the sample moments, let us denote by $\mu_{x_1,\ldots,x_p}^{m_1,\ldots,m_p}$, $\mu_{y_1,\ldots,y_q}^{r_1,\ldots,r_q}$ the multivariate moments of the variables $(x_1,\ldots,x_p)$, $(y_1,\ldots,y_p)$, of order $(m_1,\ldots,m_p)$ and $(r_1,\ldots,r_q)$, respectively. Let

$$\widehat{\mu}_{x_1,\ldots,x_p}^{m_1,\ldots,m_p} \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i=0}^{N-1} x_1^{m_1}(i)\ldots x_p^{m_p}(i)$$

be the sample estimate of $\mu_{x_1,\ldots,x_p}^{m_1,\ldots,m_p}$ drawn from N observations. An analogous definition holds for $\widehat{\mu}_{y_1,\ldots,y_q}^{r_1,\ldots,r_q}$. The generic covariance matrix element is

$$\begin{aligned} &\operatorname{cov}(\widehat{\mu}_{x_1,\ldots,x_p}^{m_1,\ldots,m_p}, \widehat{\mu}_{y_1,\ldots,y_q}^{r_1,\ldots,r_q}) \\ &= E\{\widehat{\mu}_{x_1,\ldots,x_p}^{m_1,\ldots,m_p} \cdot \widehat{\mu}_{y_1,\ldots,y_q}^{r_1,\ldots,r_q}\} - \mu_{x_1,\ldots,x_p}^{m_1,\ldots,m_p} \cdot \mu_{y_1,\ldots,y_q}^{r_1,\ldots,r_q} \\ &= \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} E\{x_1^{m_1}(i)\ldots x_p^{m_p}(i)\cdot y_1^{r_1}(j)\ldots y_q^{r_q}(j)\} \\ &\qquad - \mu_{x_1,\ldots,x_p}^{m_1,\ldots,m_p} \cdot \mu_{y_1,\ldots,y_q}^{r_1,\ldots,r_q} \end{aligned}$$

[1] We will denote by $\kappa_{xy}^{(m,n)} = CUM(\underbrace{x,\cdots,x}_{m},\underbrace{y,\cdots,y}_{n})$.

If the sample is drawn from a stationary process, the expected value of this product depends only on $k = i - j$, and the above expression simplifies as follows

$$\begin{aligned} \operatorname{cov}(\widehat{\mu}_{x_1,\ldots,x_p}^{m_1,\ldots,m_p}, \widehat{\mu}_{y_1,\ldots,y_q}^{r_1,\ldots,r_q}) &= \sum_{k=-(N-1)}^{N-1}\left[\frac{(N-|k|)}{N^2}\right. \\ &\left.\cdot E\{x_1^{m_1}(0)\ldots x_p^{m_p}(0)\cdot y_1^{r_1}(k)\ldots y_q^{r_q}(k)\}\right] \\ &- \mu_{x_1,\ldots,x_p}^{m_1,\ldots,m_p} \cdot \mu_{y_1,\ldots,y_q}^{r_1,\ldots,r_q} \end{aligned}$$

Therefore, the generic element of the cumulant covariance matrix is expressed as sum of products of multivariate moments, *i.e* as a combination of the true cumulants of the originary random variables

The only approximation introduced in the estimation of the covariance of the sample cumulants is the linearization of the relationship between sample cumulants and sample moments. The linearization error is negligible when the estimation error variance is sufficiently small; then, for sufficiently high SNR, the expression of the sample cumulants covariance stands even for a sufficiently large, but finite, number of measurements N.

3. TIME DELAY ESTIMATION

Let us refer to the following continous time model of signals received at two sensors:

$$\begin{aligned} x(t) &= s(t) + w_1(t) \\ y(t) &= s(t - D) + w_2(t) \end{aligned}$$

where $x(t), y(t)$ contain two differently delayed versions of the same signal, and two measurement noises $w_1(t), w_2(t)$. that both the signal and the noises are realizations of zero mean stationary processes. The noises are Gaussian, and can be temporally and spatially correlated. Finally, the signal and the noises are statistically independent.

The equivalent discrete time model is:

$$x[n] = s[n] + w_1[n] \quad (1)$$
$$y[n] = s_D[n] + w_2[n], \quad (2)$$

where $s[n]$, $s_D[n]$ represent two sequences obtained by sampling the signal $s(t)$ and its delayed version $s(t - D)$ with sample period T.

The problem is to find an estimate $\hat{D}$ of the actual time delay D, or, equivalently, an estimate $\hat{d}$ of the differential delay d with respect to a coarse approximation $i_m T$ of the time delay D, from a finite set of samples of the signal.

Here, we consider the estimator reported in [6] that maximizes the fourth–order cumulants of the sum

$x_\tau[n] + y[n]$, where $x_\tau[n]$ denotes the sampled version of the delayed analog signal $x(t-\tau)$. Since we are dealing with discrete–time observations, we refine the coarse TDE estimator discussed in [6], by interpolating the sampled estimates of the fourth–order cumulant with a parabole and locating the vertex. In particular, let T be the sample period of the signal, and θ be the distance between the samples of the fourth–order cumulants. Let i_m correspond to the maximum of the fourth–order sample cumulants, *i.e.* $i_m T$ be a coarse estimate of the true delay D, and d be the incremental delay $d = D - i_m T$.

Let $\hat{u}, \hat{v}, \hat{w}$ be the sample fourth–order cumulants for τ equal to $i_m T + \theta, i_m T - \theta, i_m T$, respectively, *i.e.*

$$\begin{aligned}\hat{u} &= \widehat{CUM}_4(x_{i_m T+\theta}[n] + y[n])\\ \hat{v} &= \widehat{CUM}_4(x_{i_m T-\theta}[n] + y[n])\\ \hat{w} &= \widehat{CUM}_4(x_{i_m T}[n] + y[n]).\end{aligned}$$

where $\widehat{CUM}$ denotes sample cumulant. Then, by fitting the fourth–order cumulants with the parabole through $\hat{u}, \hat{v}, \hat{w}$, we obtain the following TDE

$$\hat{d} = -\frac{\theta}{2} \cdot \frac{\hat{u} - \hat{v}}{\hat{u} - 2\hat{w} + \hat{v}}$$

Following the guidelines of [7], we can express the bias and the variance of the estimated incremental delay $\hat{d}$ in terms of the expected values U, V, W and covariances $\kappa_{uv}^{(1,1)}$ of the statistics $\hat{u}, \hat{v}, \hat{w}$ (see Eqs. (17)-(21) in [7]), that can be written in terms of cumulants of the signals involved. For simplicity, in the following we will refer to integer delay $D = i_m T$ and will assume $\theta = T = 1$. Whenever $s(n)$ is stationary and can be modeled as a signal obtained by driving a linear system $h(n)$ with an i.i.d. sequence $s_0(n)$, the expected values of the statistics are:

$$\begin{aligned}U &= \kappa_{s_0}^{(4)} \cdot \epsilon_{h(n)+h(n-1)}^{(4)},\\ V &= \kappa_{s_0}^{(4)} \cdot \epsilon_{h(n)+h(n+1)}^{(4)},\\ W &= \kappa_{s_0}^{(4)} \cdot \epsilon_{2h(n)}^{(4)},\end{aligned}$$

where

$$\epsilon_f^{(p)} = \sum_{n=-\infty}^{+\infty} f(n)^p.$$

Following [6], we model the noise spatial correlation by means of a linear filter so that

$$w_2[n] = h_{sc}[n] * w_1[n], \tag{3}$$

and the covariance between $w_1[n]$ and $w_2[n]$ is

$$\kappa_{w_1 w_2}^{(1,1)}[k] = \kappa_{w_1}^{(1,1)}[k] * h_{sc}^*[-k].$$

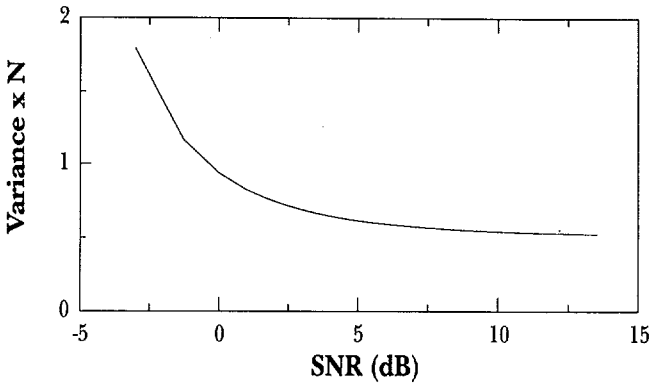

Figure 1: *White Signal and Spatially Perfectly Correlated White Noises: $N \cdot Var$ vs. SNR.*

Obviously, when $w_1[n]$ and $w_2[n]$ are spatially uncorrelated $h_{sc}[k] = 0$, while when they are perfectly spatially correlated $h_{sc}[k] = \delta[k]$. Under these hypotheses, the generic element of the covariance matrix of the sample cumulants is:

$$\begin{aligned}\kappa_{uv}^{(1,1)} = \sum_{k=-(N-1)}^{N-1} \frac{(N-|k|)}{N^2} \Big[& 24\,\kappa_{\tilde{u}\tilde{v}}^{(1,1)^4}\\ & + 144\,\kappa_{\tilde{u}\tilde{v}}^{(1,1)}\,\kappa_{\tilde{u}\tilde{v}}^{(1,2)}\,\kappa_{\tilde{u}\tilde{v}}^{(2,1)}\\ & + 72\,\kappa_{\tilde{u}\tilde{v}}^{(1,1)^2}\,\kappa_{\tilde{u}\tilde{v}}^{(2,2)} + 18\,\kappa_{\tilde{u}\tilde{v}}^{(2,2)^2}\\ & + 24\,\kappa_{\tilde{u}\tilde{v}}^{(2,1)}\,\kappa_{\tilde{u}\tilde{v}}^{(2,3)} + 16\,\kappa_{\tilde{u}\tilde{v}}^{(1,3)}\,\kappa_{\tilde{u}\tilde{v}}^{(3,1)}\\ & + 24\,\kappa_{\tilde{u}\tilde{v}}^{(1,2)}\,\kappa_{\tilde{u}\tilde{v}}^{(3,2)} + 16\,\kappa_{\tilde{u}\tilde{v}}^{(1,1)}\,\kappa_{\tilde{u}\tilde{v}}^{(3,3)}\\ & + \kappa_{\tilde{u}\tilde{v}}^{(4,4)} \Big]\end{aligned} \tag{4}$$

where

$$\begin{aligned}\kappa_{\tilde{u}\tilde{v}}^{(p,q)} &= \kappa_{x_{\tau_u}+y, x_{\tau_v}+y}^{(p,q)}\\ &= \kappa_{s_{\tau_u}+s_D, s_{\tau_v}+s_D}^{(p,q)} + \kappa_{w1_{\tau_u}+w2, w1_{\tau_v}+w2}^{(p,q)}\end{aligned}$$

For the sake of the comprehension, we report here the expression of the TDE variance for $d = 0$

$$\mathrm{var}(\hat{d}) = \frac{\theta}{16} \cdot \frac{1}{(U-W)^2} \cdot \left(\sigma_u^2 - 2\kappa_{uv}^{(1,1)} + \sigma_v^2\right)$$

where $\sigma_u^2 = \kappa_{uu}^{(1,1)}$ and $\sigma_v^2 = \kappa_{vv}^{(1,1)}$ are the variance of $\hat{u}$ and $\hat{v}$, respectively. For visual reference, we report here the asymptotic variances, and biases, of the TDE estimator, for an i.i.d. zero–mean one–sided exponential sequence $s_0(n)$. In this case, the univariate cumulant for the signal distribution, is

$$\kappa_s^{(r)} = \sigma_s^r \cdot \epsilon_{h(n)}^{(r)} \cdot (r-1)!. \tag{5}$$

In Fig.(1), signals and noise are white; noises are perfectly spatially correlated. The bias is zero and, interestingly

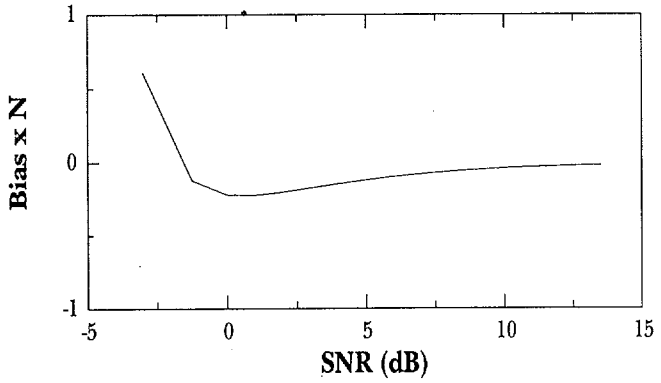

Figure 2: *Coloured Signal and Spatially Correlated Coloured Noises: $N \cdot Bias$ vs SNR. The colouring filter is low-pass with cutoff $f_c = 0.75\pi$.*

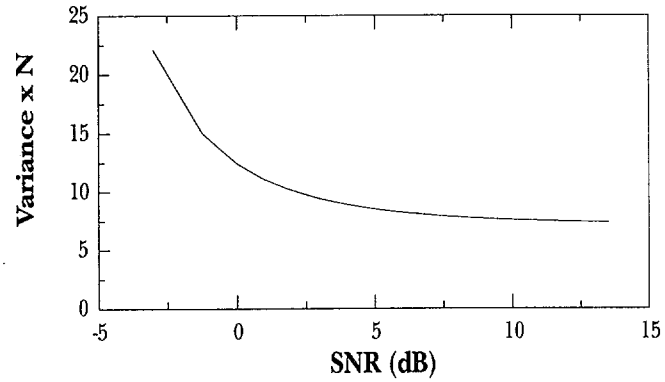

Figure 3: *Coloured Signal and Spatially Correlated Coloured Noises: $N \cdot Var$ vs. SNR. The colouring filter is low-pass with cutoff $f_c = 0.75\pi$.*

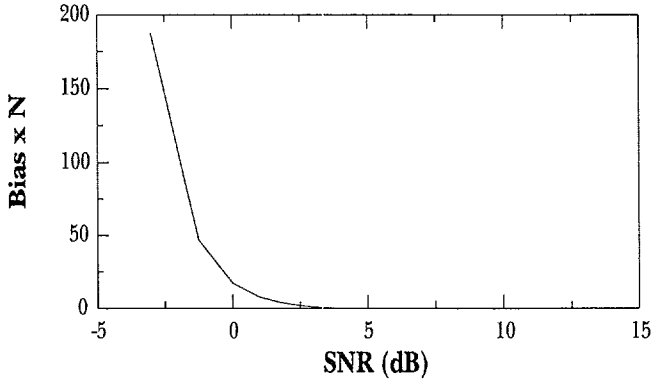

Figure 4: *Coloured Signal and Spatially Correlated Coloured Noises: $N \cdot Bias$ vs. SNR. The colouring filter is low-pass with cutoff $f_c = 0.48\pi$.*

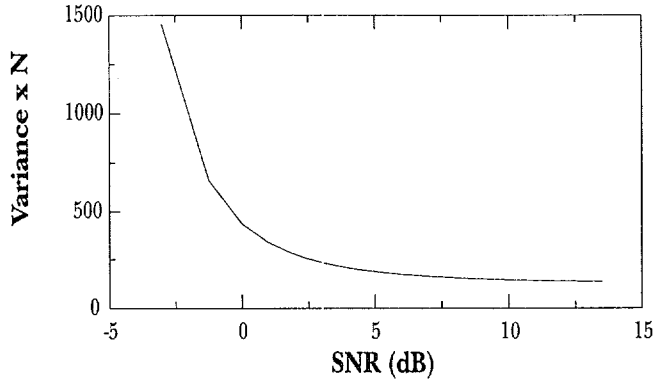

Figure 5: *Coloured Signal and Spatially Correlated Coloured Noises: $N \cdot Var$ vs. SNR. The colouring filter is low-pass with cutoff $f_c = 0.48\pi$.*

enough, it can be seen from the formulas that the variance turns out to be the same also if the noises are spatially uncorrelated.

In Figs.2–3, signal and noise are low-pass filtered with cutoff frequency $f_c = 0.75\pi$, while in Figs.4–5 the cutoff frequency is $f_c = 0.48\pi$; a ten taps filter models the spatial correlation between the noises (see [6]).

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