Can Hurst Analysis and Surrogate Data Methods Form a Reliable Test for Chaos Detection?

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RÉSUMÉ

Cet article étudie la réalisation d'un test qualitatif sur le chaos a partir de tests statistiques aussi simples que possible. Deux tests complémentaires sont utilisés, l'un testant la fractalité, l'autre la nonlinéarité d'un signal. Pour chacun de ces tests, des options sont proposées et leurs performances discutées. Chacune est ameliorée chaque fois que c'est possible en utilisant en complément les données de substitution, qui donnent une information sur la validite des résultats. Ces tests se révèlent intéressants, mais une recherche ultérieure est nécessaire afin d'améliorer encore leur rapidité et leur fiabilité.

ABSTRACT

This paper considers the possibility of making a qualitative test on chaos from simple statistical tests. Two complementary tests are used for detecting chaos, one testing the data for fractality, the other one for nonlinearity. Different options are proposed for each test, and their performances are discussed. Each option is improved whenever possible by using the surrogate data method, which gives informations on the significance of the results obtained. These tests have shown to be interesting, but further work is necessary in order to improve their speed and reliability.

1 Introduction

Nowadays, one of the biggest problems encountered when dealing with chaos is detecting it in experimental data. All methods aimed at this end (dimension computation, Lyapunov exponents) must be used with care, and are very sensitive to many parameters, such as length of the data or noise. Besides, at least from the authors' knowledge, there exists no reliable test on chaos, far less an efficient one. The aim of this paper is to open the discussion on the subject, by proposing a new qualitative test based on statistical methods, the Hurst or R/S analysis [1, 2] and the so-called long-term correlation analysis [3] for testing the fractality of the data, and the surrogate data method proposed by Theiler in [4] for testing their nonlinearity. It is indeed very difficult to design a single test for chaos, at least from the current knowledge on the subject. There is however a way out: chaos is defined as being a deterministic nonlinear process, with self-similarity, or fractal nature. If one can find a test for nonlinearity and fractality, and put them together, the result should be a chaos test, hence the above propositions. After a brief theoretical description, each of these methods is studied in terms of performance and speed, and the best combination of fractality and nonlinearity tests will be proposed for the chaos detection procedure.

2 Fractality tests

One test on the fractal nature of a time series has been developped in 1951, by Hurst [1]. Let $\{x_k\}$ be a time series of length N. Its mean and standard deviation are x_{mn} and x_{std} respectively. One defines the cumulative series of $\{x_k\}$ with respect to the mean by

$$X_{k,N} = \sum_{i=1}^{k}(x_i - x_{mn})$$

The adjusted range is given by

$$R(N) = max_{1\leq k\leq N}[X(k,N)] - min_{1\leq k\leq N}[X(k,N)]$$

The rescaled range is then

$$\frac{R(N)}{x_{std}} = \frac{R}{S}$$

Hurst discovered that, in the case of a fractal series, the rescaled range obeys the power law

$$\frac{R}{S} = c \cdot N^H$$

where H is called the Hurst exponent. It thus suffices to be able to compute the rescaled range R/S to obtain the Hurst exponent H, which gives a measure of the fractality of the time series. This computation is called the Hurst or rescaled range analysis. H is in the interval between 0 and 1. When H is equal to 1/2, the series is a Gaussian process, and when it is significantly higher than 1/2, the series is fractal.

The Hurst analysis was tested on different signals, i.e. Gaussian noise, fractional noise, a simple nonlinear process and the Henon map, for two different lengths of the data, 15000 and 5000 samples. The results are shown in the following table:

	H (15000 smp)	H (5000 smp)
gaussian n.	0.45	0.4
fractal n.	0.95	0.71
nonlin. proc.	0.61	0.9
henon	0.82	0.28

If the results of H with 15000 samples data are relatively good (the expected value for the Gaussian noise is 0.5, and 0.92 for Henon, for instance) it is not the case when the data length is shorter. In the latter case, the results cannot be trusted anymore. The large difference between the theoretical and experimental results comes from the fact that the Hurst equation is an asymptotical one. When the time series used are of finite length, there are deviations from the theoretical values, and the deviations are inversely proportional to the length of the data.

Since most real-world data are of short length, the Hurst analysis cannot be trusted as a fractality test. Hence the idea of another test, proposed by Buldyrev *et al* in [3]. This test, called the long-term correlation analysis, examines the long-term correlations in a signal, just as the Hurst analysis does. Anyway, it is more simple and allows to have short time series as input. First developed for the analysis of DNA sequences [3], it can be easily extended to arbitrary time series. The procedure therefore consists in first computing the difference function $d(l)$

$$d(l) = y(l_0 + l) - y(l_0)$$

where $y(l)$ is the original time series, then finding the mean square fluctuation function $F(l)$

$$F^2(l) = \overline{d^2(l)} - \overline{d(l)}^2$$

Where the horizontal bar stands for the average. Here, the average is computed according to different values of l_0. Three cases may occur: If $F(l) \sim e^{0.5}$, the signal is random, this is a random walk. If $F(l) \sim e^{-l/R}$, there are correlations extending up to a characteristic range R, but the asymptotic behaviour is unchanged from the purely random case. At last, if $F(l) \sim l^{\alpha}$, there are 'infinite-range' correlations and the signal is fractal, if $\alpha \neq 0.5$. These properties can be observed on the shape of the loglog plot of $F(l)$: if the signal is random, $log(F(l))$ looks like a straight horizontal line or a random curve. If the signal si fractal, it looks like a straight line, with a slope different from $\alpha = 0.5$.

The typical results obtained for a fractal or chaotic signal are shown on fig 1, where one can see the $log(F(l))$ plot.

The curve obtained is not exactly a straight line because there is saturation after some point, This probably comes from the fact that the time series is of finite length: the long-term correlations can exist only up to a certain range, due to the short length of the data (5000 samples). Nevertheless, one can clearly see the difference between this curve and the ones corresponding to non-fractal signals (fig. 2.a is the result for a Gaussian noise, fig. 2.b is the result for a nonlinear process).

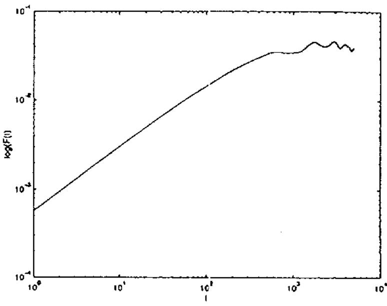

Figure 1: Long-term correlation analysis for a fractional noise

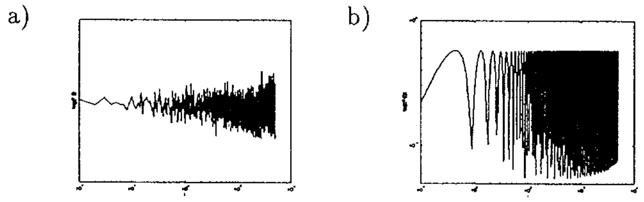

Figure 2: a) Fractal landscape and b) local slope for a Gaussian noise

In the same way, on can see by comparing figures 3 and 1 that this procedure is unable to discriminate between a fractional noise and a chaotic signal.Besides, because of the finite length of the data, it can confuse actual fractal signals with signals having correlations up to a certain caracteristic range. But it will not confuse a fractal or chaotic signal with a non-fractal one.

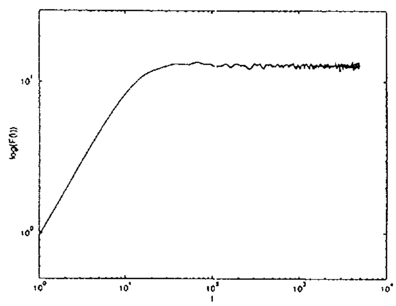

Figure 3: Long-term correlation analysis for the Lorenz system

This test can be improved by using surrogate data, so that one can check the significance of the results. It consists in testing a time series by confronting the data to a null hypothesis. In a few words, one specifies the null hypothesis for the process under study, then generates surrogate data sets that share most of the properties of the original time series, but are consistent with the hypothesis. Discriminating statistics are then computed to compare actual and surrogate data. If the difference between the statistics is significant, the null hypothesis is rejected. The significance is computed by

$$SIG(l) = \frac{|F(l) - F_{mn}(l)|}{F_{std}(l)}$$

where $F(l)$ is the value of the statistics for the actual data, $F_{mn}(l)$ the mean of the statistics for the surrogate data, while $F_{std}(l)$ is its standard deviation. The significance of the long-term correlation analysis results for a fractional noise are displayed in figure 4.a, while figure 4.b shows the same parameter for the Gaussian noise.

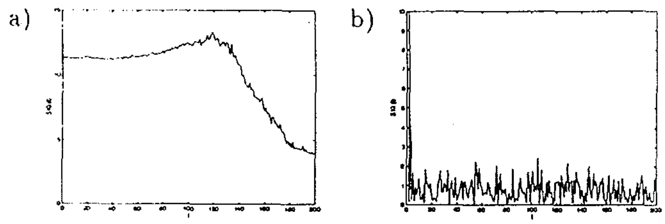

Figure 4: Significance *vs* l for a) fractional noise, b) Gaussian noise)

This shows clearly that one can have confidence in the results obtained by the long-term correlation analysis, since the significance is very high for the fractional noise, and not for the Gaussian noise. In summary, one can say that this test is much more reliable than the Hurst analysis, since it discriminates correctly between the signals for much shorter data. Even if it can be flawed in some cases, it is not an important defect, since it can misclassify non-fractal signals as fractal ones, and not the opposite. And our interest in is a preliminary test, not obligatorily a decisive one.

3 Nonlinearity tests

A suitable method for testing nonlinearity is the surrogate data method proposed by Theiler in [4]. It consists in testing a time series by confronting the data to a null hypothesis. For testing nonlinearity, several null hypotheses need to be considered. The procedure can be separated into 3 steps: first, specifying a null hypothesis to which the data must be confronted. In the second step, surrogate data are generated, which share the same properties as the actual data, except the one necessary to make them consistent with the hypothesis. At last, a discriminating statistics is computed for the actual and surrogate data, and the results are compared. If the statistics has/have significantly different values for the signal and its surrogates, the null hypothesis can be rejected. As chaos is the subject of interest here, the discriminating statistics used are the local intrinsic dimension (LID) [5], and the computation of the largest Lyapunov exponent [6].Three different null hypotheses were chosen here: first, the hypothesis of a IID noise with arbitrary amplitude distribution. Then the one of a linear process, and last a nonlinear transform of a linear gaussian noise. If these three hypotheses are rejected, one can conclude that the process under study is nonlinear. The surrogate data corresponding to each proposition were computed as proposed in [4]. The significance is computed in the same way as for the long-term correlation analysis.

This method is well-known, which makes it unnecessary to show complete results. The results given here will therefore be only thoses relevant to the aims of this work, namely the discrimination between fractional noise and chaotic signals, which the fractality tests confuse. When applied to a fractional noise, the Local Intrinsic Dimension computation gives the results of table 3 for the null hypothesis of arbitrary noise, and the results of table 3 for the linear dynamics hypothesis, each for the fractional noise and the Henon map:

	Henon	fractional
LID	2.1	1.15
mean(LID)	1.00	1.00
difference	110%	15%
significance	60.8	38.51

	Henon	fractional
LID	2.1	1.15
mean(LID)	1.0036	1.0064
difference	109.5%	14.4%
significance	85.3	2.1

In each table, *LID* stands for Local Intrinsic Dimension obtained for the actual data, *mean(LID)* for the mean LID of the surrogate data. *Difference* stands for the difference between the two preceding results, while *significance* is the significance of this difference.

It is clear from both tables that the Henon map is chaotic, since the difference between the actual LID and the surrogates mean LID is important and significant. On the contrary, the fractional noise is correctly identified as being noise, since the difference and significance are low for the test against noise. The difference remains low between the actual and mean LID in the case of the linear dynamics tests, but since the significance is equaly low, the result is irrelevant. The Maximum Lyapunov Exponent (MLE) computation results can be seen on the following figures:

The curves displayed on figures 5.a and 5.b show the average distance between neighboring points according with respect to time. Since the method supposes that the largest Lyapunov exponent is positive, the curves have a positive slope whatever the signal. The chaotic signal can be recognised in that the curve doesn't vary for different reconstruction dimensions. The slope of the linear part of the curve gives the Lyapunov exponent. One can see that in both cases the mean curves of the surrogate data (dashed line for the noise test, dotted line for the linear dynamics one) are clearly different from the actual one (full line). But for Henon, the significance is very high for both hypothesis, while it is low for the fractional noise, when confronted with the noise hypothesis.

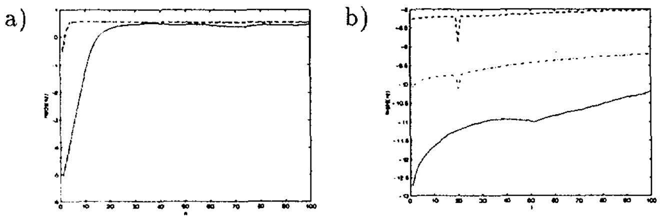

Figure 5: Lyapunov calculation results for a) the Henon map, b) the fractional noise. (full line: actual values, dashed line: mean value for the noise test, dotted line: mean value for the linear dynamics test

Theiler's surrogate data method is known as being one of the most reliable test for nonlinearity in possible chaotic signals. From our point of view, it nevertheless suffers from some very important drawbacks. It is in a sense redundant with its own statistics, since both LID and MLE can be rather reliable tests for chaos, as was shown in [7]. The surrogate data test is then not necessary. Besides, Theiler's method suffers from an important flaw, with respect to the goal of this paper: it uses chaos tools for detecting chaos, when the goal was to avoid this use, since using them means making some *a priori* hypotheses on the nature of the signal. Finally, its computation load is too high for a simple fast qualitative test.

Very recently, a method has been proposed in [8], which could be a good alternative to the surrogate data procedure. This method tests for nonlinearity by comparing the redundancy and the linear redundancy *vs* the lag for different reconstruction dimensions. If both families of curves have the same shape, the signal under study is linear; in the opposite case, it is a nonlinear dynamical system. This procedure has the advantage of using no chaotic tool, only statistics ones, and seems, according to preliminary tests, to be much faster than Theiler's. However, it remains to be confirmed that it discriminates correctly between stochastic or deterministic nonlinear processes and other types of processes.

4 Discussion

It has been shown that it is possible to detect chaos while avoiding as much as possible the use of chaos oriented tools. Some recent publications [8] let think that detecting chaos without using any chaos oriented tool is possible. Besides, the results presented here show that the basic idea behind this work is valid, e.g. that two complementary tests, one on fractality, the other on nonlinearity, help detect chaos in time series, with no *a priori* knowledge of their properties. The fractality test allows the discrimination between fractal and non-fractal processes, thus between nonlinear processes and chaos. The nonlinarity test, for its part, allows a correct discrimination between fractional noise and chaos, which the fractality test cannot do. Of course, both classify correctly non-fractal noise, deterministic or stochastic linear processes as not being chaotic.

In what concerns the choice of the proper set of tests, it is obvious that the best fractality test is the long-term correlation analysis. The classical Hurst analysis is indeed unreliable as soon as the data consist of less than 10000 samples, which is often the case in experimental situations. Concerning the nonlinearity test, the reliability of Theiler's method is well known, but it suffers from a heavy computation load. If the redundancy method is demonstrated to be reasonably reliable, it should be prefered to the former: The goal is not, at least for now, to have a 100% reliable test, but to have a fast qualitative test that can help decide if it is worth studying the chaotic aspect of a signal or not.

However, these tests are not perfect yet. They need further study to assess their robustness with respect to noise or length of the data, and they suffer from flaws that should be removed. For instance, the long-term correlation method can, as was said before, misinterpret some signals with long-term correlations up to a given range as infinite-range correlations ones, and thus classify the former as being fractal. Besides, it would be interesting to develop innovative tests that would be faster and/or more reliable. Since the goal of this paper was to present essentially qualitative tests, finding a numerical parameter was not a priority, but this would be a important progress in the search for a reliable test. A first step was made in using the surrogate data tests whenever possible, but it still doesn't remove all possibility of misinterpretation due to the user's subjectivity.

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