EFFECTS OF NONRANDOM LINEAR TIME-VARIANT SYSTEMS ON HIGHER-ORDER CYCLOSTATIONARITY

Luciano IZZO **Antonio NAPOLITANO**

Università di Napoli Federico II, Dipartimento di Ingegneria Elettronica
via Claudio 21, I-80125 Napoli, Italy, E-mail: izzo@nadis.dis.unina.it

RÉSUMÉ

Nous proposons dans cet article une contribution à la théorie de la cyclostationnarité d'ordre supérieur introduite récemment pour généraliser la théorie de la cyclostationnarité d'ordre deux. Nous étudions les effets sur les statistiques cycliques d'ordre supérieur à deux de ces systèmes linéaires variant dans le temps qui transforment une série temporelle presque périodique dans une autre série temporelle presque périodique. La classe considérée comprend les systèmes linéaires variant dans le temps presque périodiquement et par conséquent les systèmes lineaires invariant dans le temps.

ABSTRACT

A contribution to the theory of higher-order cyclostationarity very recently introduced to generalize the second-order cyclostationarity is given. The effects on the cyclic higher-order statistics of those linear time-variant systems that map almost-periodic inputs into almost-periodic outputs are investigated. The class of systems considered in the paper includes the linear almost-periodically time-variant systems and hence the linear time-invariant systems.

1. INTRODUCTION

In recent years, the theory of signals from which finite-strength additive sinewaves can be generated by using quadratic transformations has been developed. Moreover, such a theory has been applied to problems of weak-signal detection, parameter estimation, system identification, etc. [1]. Algorithms based on spectral line generation are asymptotically independent of both noise and interference and then result to be highly tolerant to noise and interference in practice.

There are some signals from which spectral lines cannot be generated by a quadratic transformation, but from which spectral lines can be generated by using an Nth-order nonlinear transformation with $N > 2$. The minimum order of nonlinearity that is necessary to generate a sinewave from a signal is called the order of cyclostationarity of the signal, and the frequency of the regenerated sinewave is called cycle frequency. For example, a pulse-amplitude-modulated signal with bandwidth equal to the Nyquist rate is fourth-order cyclostationary because no nonlinearities of order less than four can generate a sinewave from such a signal, but a fourth-order nonlinear transformation can generate a sinewave with frequency equal to the pulse rate [2],[3].

Relevant results on the theory of higher-order wide-sense cyclostationarity (WSCS) have been very recently presented in both time-series and stochastic process frameworks. Moreover, applications to higher-order spectrum estimation, system identification, and weak-signal detection have been presented (see [2]-[6] and references therein).

The present paper investigates the way in which the higher-order WSCS properties of time-series change as they are processed by linear time-variant systems belonging to the class of "stationary" systems introduced by Claasen and Mecklenbräuker in [7]. This class of systems is that which maps almost-periodic inputs into almost-periodic outputs. Therefore, in the fraction-of-time (FOT) probability framework it can be referred to as the class of nonrandom systems. Linear almost-periodically time-variant systems and hence linear time-invariant systems are subclasses of the class considered here.

2. BACKGROUND

In this section the class of nonrandom linear time-variant systems is presented and a brief introduction on higher-order wide-sense cyclostationarity is provided.

A. Nonrandom linear time-variant systems

In the fraction-of-time probability context, a nonrandom system is a possibly complex (and not necessarily linear) system that for every deterministic (i.e., constant, periodic, or polyperiodic) input time-series delivers a deterministic output time-series. Therefore, for a system input time-series

$$x(t) = e^{j2\pi\lambda t}, \tag{1}$$

the system output time-series $y(t)$ can be expressed as

$$y(t) = \sum_{\sigma\in\Omega} G_\sigma(\lambda) e^{j2\pi\varphi_\sigma(\lambda)t}, \tag{2}$$

where Ω is a finite or denumerable set and $G_\sigma(\cdot)$ and $\varphi_\sigma(\cdot)$ are complex functions and monotonic real functions (respectively) that characterize the system.

A nonrandom linear time-variant system can be characterized by the transmission function [7]

$$H(f,\lambda) = \sum_{\sigma\in\Omega} G_\sigma(\lambda)\,\delta(f-\varphi_\sigma(\lambda)) = \sum_{\sigma\in\Omega} H_\sigma(f)\,\delta(\lambda-\psi_\sigma(f)), \tag{3}$$

where $\delta(\cdot)$ is Dirac's delta function, the functions $\psi_\sigma(\cdot)$, referred to as the frequency mapping functions, are the inverse functions of $\varphi_\sigma(\cdot)$, and

$$H_\sigma(f) \triangleq |\dot{\psi}_\sigma(f)|\,G_\sigma(\psi_\sigma(f)), \tag{4}$$

in which $\dot{\psi}_\sigma(\cdot)$ denotes the derivative of the function $\psi_\sigma(\cdot)$.

By inverse Fourier transforming the first and the last sides of (3), one obtains the expression of the impulse-response function

$$h(t,u) = \sum_{\sigma\in\Omega} h_\sigma(t)\otimes\Psi_\sigma(t,u), \tag{5}$$

where $\otimes$ denotes convolution, $h_\sigma(t)$ is the inverse Fourier transform of $H_\sigma(f)$, and

$$\Psi_\sigma(t,u) \triangleq \int_{-\infty}^{+\infty} e^{-j2\pi\psi_\sigma(f)u} e^{j2\pi ft}\,df. \tag{6}$$

The class of nonrandom linear time-variant systems includes that of the linear almost-periodically time-variant (LAPTV) systems which, in turn, includes, as special cases, both linear periodically time-variant and linear time-invariant systems. For the LAPTV systems, the frequency mapping functions $\psi_\sigma(f)$ are linear with unitary slope, that is,

$$\psi_\sigma(f) = f-\sigma, \qquad \sigma\in\Omega, \tag{7}$$

and then the impulse-response function can be expressed as

$$h(t,u) = \sum_{\sigma\in\Omega} h_\sigma(t-u)e^{j2\pi\sigma u}. \tag{8}$$

The systems performing time scale changing belong to the class under consideration. In such a case, the impulse-response function is given by

$$h(t,u) = \delta(u-at), \tag{9}$$

where $a \neq 0$ is the scale factor, the set Ω contains just one element, and

$$\psi_\sigma(f) = \frac{f}{a}, \qquad H_\sigma(f) = \frac{1}{|a|}. \tag{10}$$

Linear time-variant systems that cannot be modeled as nonrandom include chirp modulators, modulators whose carrier frequency is a pseudo-noise sequence (as in the spread-spectrum modulation), and systems performing time windowing.

B. Higher-order cyclostationarity

In the FOT probability context, a possibly complex-valued time-series $x(t)$ is said to exhibit Nth-order wide-sense cyclostationarity with cycle frequency $\alpha \neq 0$ if at least one of the Nth-order cyclic temporal moment functions (CTMF's)

$$\mathcal{R}^\alpha_{\boldsymbol{x}}(\boldsymbol{\tau})_N \triangleq \left\langle \prod_{n=1}^{N} x^{(*)_n}(t+\tau_n)e^{-j2\pi\alpha t}\right\rangle \tag{11}$$

is not identically zero. In (11), $\boldsymbol{\tau} \triangleq [\tau_1,\dots,\tau_N]^{\mathrm{T}}$ and $\boldsymbol{x} \triangleq [x^{(*)_1}(t),\dots,x^{(*)_N}(t)]^{\mathrm{T}}$ are column vectors, $\langle\cdot\rangle$ denotes infinite time averaging, and $(*)_n$ represents optional conjugation of the nth factor of the lag product $\prod_{n=1}^{N} x^{(*)_n}(t+\tau_n)$. The magnitude and phase of the function $\mathcal{R}^\alpha_{\boldsymbol{x}}(\boldsymbol{\tau})_N$ are amplitude and phase of the sinewave component with frequency α contained in the lag product.

If the set of Nth-order cycle frequencies, say $A_{x,N}$, is finite or denumerable, the time-series $x(t)$ is said to be wide-sense almost-cyclostationary [2]. In such a case, the expected value of the lag product, which is called the Nth-order temporal moment function, is defined by

$$\mathcal{R}_{\boldsymbol{x}}(t,\boldsymbol{\tau})_N \triangleq \sum_{\alpha\in A_{x,N}} \mathcal{R}^\alpha_{\boldsymbol{x}}(\boldsymbol{\tau})_N e^{j2\pi\alpha t}. \tag{12}$$

The N-fold Fourier transform $\mathcal{S}^\alpha_{\boldsymbol{x}}(\boldsymbol{f})_N$ of the CTMF is called the Nth-order cyclic spectral moment function (CSMF) and can be written as [2]

$$\mathcal{S}^\alpha_{\boldsymbol{x}}(\boldsymbol{f})_N = S^\alpha_{\boldsymbol{x}}(\boldsymbol{f}')_N\delta(\boldsymbol{f}^{\mathrm{T}}\mathbf{1}-\alpha), \tag{13}$$

where $\boldsymbol{f} \triangleq [f_1,\dots,f_N]^{\mathrm{T}}$, $\mathbf{1} \triangleq [1,\dots,1]^{\mathrm{T}}$, and prime denotes the operator that transforms a vector $\boldsymbol{w} \triangleq [w_1,\dots,w_K]^{\mathrm{T}}$ into $\boldsymbol{w}' \triangleq [w_1,\dots,w_{K-1}]^{\mathrm{T}}$. The function $S^\alpha_{\boldsymbol{x}}(\boldsymbol{f}')_N$, referred to as the Nth-order reduced-dimension CSMF (RD-CSMF), can be expressed as the $(N-1)$-fold Fourier transform of the Nth-order reduced-dimension CTMF (RD-CTMF) defined as

$$R^\alpha_{\boldsymbol{x}}(\boldsymbol{\tau}')_N \triangleq \mathcal{R}^\alpha_{\boldsymbol{x}}(\boldsymbol{\tau})_N\,|_{\tau_N=0}\,. \tag{14}$$

Let us now consider the Nth-order temporal cumulant function

$$\mathcal{C}_{\boldsymbol{x}}(t,\boldsymbol{\tau})_N = \sum_{\mathsf{P}}\left[(-1)^{p-1}(p-1)!\prod_{i=1}^{p}\mathcal{R}_{\boldsymbol{x}_{\mu_i}}(t,\boldsymbol{\tau}_{\mu_i})_{|\mu_i|}\right], \tag{15}$$

where P is the set of distinct partitions of $\{1,\dots,N\}$, each constituted by the subsets $\{\mu_i:\ i=1,\dots,p\}$, $|\mu_i|$ is the number of elements in μ_i, and $\boldsymbol{x}_{\mu_i}$ is the $|\mu_i|$-dimensional vector whose components are those (possibly conjugate) of $\boldsymbol{x}$ having indices in μ_i. Taking the N-dimensional Fourier transform of the coefficient of the Fourier series expansion of the almost-periodic function $\mathcal{C}_{\boldsymbol{x}}(t,\boldsymbol{\tau})_N$,

$$\mathcal{C}^\beta_{\boldsymbol{x}}(\boldsymbol{\tau})_N \triangleq \left\langle \mathcal{C}_{\boldsymbol{x}}(t,\boldsymbol{\tau})_N e^{-j2\pi\beta t}\right\rangle, \tag{16}$$

which is referred to as the Nth-order cyclic temporal cumulant function (CTCF), one obtains the Nth-order cyclic spectral cumulant function $\mathcal{P}^\beta_{\boldsymbol{x}}(\boldsymbol{f})_N$. It can be written as [2]

$$\mathcal{P}^\beta_{\boldsymbol{x}}(\boldsymbol{f})_N = P^\beta_{\boldsymbol{x}}(\boldsymbol{f}')_N\delta(\boldsymbol{f}^{\mathrm{T}}\mathbf{1}-\beta), \tag{17}$$

where the Nth-order cyclic polyspectrum $P_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}')_N$ is the $(N-1)$-dimensional Fourier transform of

$$C_{\boldsymbol{x}}^{\beta}(\boldsymbol{\tau}')_N \triangleq \mathcal{C}_{\boldsymbol{x}}^{\beta}(\boldsymbol{\tau})_N \,|_{\tau_N=0}, \tag{18}$$

which is the reduced-dimension CTCF.

3. INPUT/OUTPUT RELATIONS FOR NONRANDOM LINEAR SYSTEMS

Let us consider a nonrandom linear time-variant system excited by a wide-sense almost-cyclostationary time-series $x(t)$ whose set of Nth-order cycle frequencies, for the considered conjugation configuration, is $B_{x,N}$.

The Nth-order CTMF at the cycle frequency α of the output time-series $y(t)$ can be derived accounting for (5) and (11):

$$\mathcal{R}_{\boldsymbol{y}}^{\alpha}(\boldsymbol{\tau})_N = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \sum_{\boldsymbol{\sigma}\in\Omega^N} \left(\prod_{n=1}^{N} h_{\sigma_n}^{(*)_n}(u_n)\right) \cdot \sum_{\beta\in B_{x,N}} \mathcal{R}_{\boldsymbol{x}}^{\beta}(\boldsymbol{v})_N \mathcal{R}^{\alpha-\beta}_{\Psi_{\sigma_1}^{(*)_1},\ldots,\Psi_{\sigma_N}^{(*)_N}}(\boldsymbol{\tau}-\boldsymbol{u},\boldsymbol{v})_N \,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{v}, \tag{19}$$

where

$$\mathcal{R}^{\gamma}_{\Psi_{\sigma_1}^{(*)_1},\ldots,\Psi_{\sigma_N}^{(*)_N}}(\boldsymbol{\tau},\boldsymbol{v})_N \triangleq \left\langle \prod_{n=1}^{N} \Psi_{\sigma_n}^{(*)_n}(t+\tau_n, t+v_n) e^{-j2\pi\gamma t}\right\rangle. \tag{20}$$

Moreover, the input/output relation in terms of RD-CTMF's can be easily obtained setting $\tau_N = 0$ in (19).

Taking the N-dimensional Fourier transform of both sides of (19), one obtains the input/output relation in terms of CSMF's:

$$\mathcal{S}_{\boldsymbol{y}}^{\alpha}(\boldsymbol{f})_N = \sum_{\boldsymbol{\sigma}\in\Omega^N} \left(\prod_{n=1}^{N} H_{\sigma_n}^{(*)_n}((-)_n f_n)\right) \cdot \sum_{\beta\in B_{x,N}} \mathcal{S}_{\boldsymbol{x}}^{\beta}\left(\boldsymbol{\psi}_{\boldsymbol{\sigma}}^{(-)}(\boldsymbol{f}^{(-)})\right)_N \delta_{\alpha-\beta+\boldsymbol{\psi}_{\boldsymbol{\sigma}}^{(-)}(\boldsymbol{f}^{(-)})^{\mathrm{T}}\mathbf{1}-\boldsymbol{f}^{\mathrm{T}}\mathbf{1}}, \tag{21}$$

where $(-)_n$ denotes an optional minus sign to be considered only when the optional conjugation $(*)_n$ is present, $\boldsymbol{f}^{(-)} \triangleq [(-)_1 f_1, \ldots, (-)_N f_N]^{\mathrm{T}}$, $\boldsymbol{\psi}_{\boldsymbol{\sigma}}^{(-)}(\boldsymbol{f}) \triangleq [(-)_1\psi_{\sigma_1}(f_1), \ldots, (-)_N\psi_{\sigma_N}(f_N)]^{\mathrm{T}}$, and $\delta_\gamma = 1$ for $\gamma = 0$ and $\delta_\gamma = 0$ for $\gamma \neq 0$. Furthermore, from (21), accounting for (13) and the relationship

$$\delta\left(\beta - \boldsymbol{\psi}_{\boldsymbol{\sigma}}^{(-)}(\boldsymbol{f}^{(-)})^{\mathrm{T}}\mathbf{1}\right) \delta_{\alpha-\beta+\boldsymbol{\psi}_{\boldsymbol{\sigma}}^{(-)}(\boldsymbol{f}^{(-)})^{\mathrm{T}}\mathbf{1}-\boldsymbol{f}^{\mathrm{T}}\mathbf{1}} = |\dot{\varphi}_{\sigma_N}((-)_N(\beta - \boldsymbol{\psi}_{\boldsymbol{\sigma}'}^{(-)\prime}(\boldsymbol{f}^{(-)\prime})^{\mathrm{T}}\mathbf{1}))|\, \delta(\alpha - \boldsymbol{f}^{\mathrm{T}}\mathbf{1})\, \delta_{\beta-\boldsymbol{\psi}_{\boldsymbol{\sigma}}^{(-)}(\boldsymbol{f}^{(-)})^{\mathrm{T}}\mathbf{1}} \tag{22}$$

one obtains the input/output relation in terms of RD-CSMF's:

$$S_{\boldsymbol{y}}^{\alpha}(\boldsymbol{f}')_N = \sum_{\boldsymbol{\sigma}\in\Omega^N} \left(H_{\sigma_N}^{(*)_N}((-)_N(\alpha - \boldsymbol{f}'^{\mathrm{T}}\mathbf{1})) \prod_{n=1}^{N-1} H_{\sigma_n}^{(*)_n}((-)_n f_n)\right) \cdot |\dot{\varphi}_{\sigma_N}((-)_N\psi_{\sigma_N}((-)_N(\alpha - \boldsymbol{f}'^{\mathrm{T}}\mathbf{1})))| \sum_{\beta\in B_{x,N}} S_{\boldsymbol{x}}^{\beta}\left(\boldsymbol{\psi}_{\boldsymbol{\sigma}'}^{(-)\prime}(\boldsymbol{f}^{(-)\prime})\right)_N \cdot \delta_{\beta-\boldsymbol{\psi}_{\boldsymbol{\sigma}'}^{(-)\prime}(\boldsymbol{f}^{(-)\prime})^{\mathrm{T}}\mathbf{1}-(-)_N\psi_{\sigma_N}((-)_N(\alpha-\boldsymbol{f}'^{\mathrm{T}}\mathbf{1}))}. \tag{23}$$

Equation (23) shows that the RD-CSMF $S_{\boldsymbol{y}}^{\alpha}(\boldsymbol{f}')_N$ depends in general on cycle frequencies of the input time-series whose values are related to $\boldsymbol{f}'$. However, in the special case of LAPTV systems, accounting for (7), (23) reduces to

$$S_{\boldsymbol{y}}^{\alpha}(\boldsymbol{f}')_N = \sum_{\boldsymbol{\sigma}\in\Omega^N} \left(H_{\sigma_N}^{(*)_N}((-)_N(\alpha - \boldsymbol{f}'^{\mathrm{T}}\mathbf{1})) \cdot \prod_{n=1}^{N-1} H_{\sigma_n}^{(*)_n}((-)_n f_n)\right) S_{\boldsymbol{x}}^{\alpha-\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{1}}(\boldsymbol{f}' - \boldsymbol{\sigma}')_N, \tag{24}$$

and then the values of the input cycle frequencies that give a contribution to the output RD-CSMF $S_{\boldsymbol{y}}^{\alpha}(\boldsymbol{f}')_N$ are independent of $\boldsymbol{f}'$ [5],[6].

The support in the $(\alpha, \boldsymbol{f}')$ space of the RD-CSMF $S_{\boldsymbol{y}}^{\alpha}(\boldsymbol{f}')_N$ (given by (23)) can be written as

$$\mathrm{supp}\left\{S_{\boldsymbol{y}}^{\alpha}(\boldsymbol{f}')_N\right\} = \bigcup_{\boldsymbol{\sigma}\in\Omega^N} \left\{\mathrm{supp}\left\{H_{\sigma_N}^{(*)_N}((-)_N(\alpha - \boldsymbol{f}'^{\mathrm{T}}\mathbf{1})) \cdot \prod_{n=1}^{N-1} H_{\sigma_n}^{(*)_n}((-)_n f_n) |\dot{\varphi}_{\sigma_N}((-)_N\psi_{\sigma_N}((-)_N(\alpha - \boldsymbol{f}'^{\mathrm{T}}\mathbf{1})))|\right\} \bigcap \left\{\bigcup_{\beta\in B_{x,N}} \mathrm{supp}\left\{S_{\boldsymbol{x}}^{\beta}\left(\boldsymbol{\psi}_{\boldsymbol{\sigma}'}^{(-)\prime}(\boldsymbol{f}^{(-)\prime})\right)_N\right\} \bigcap \left\{(\alpha, \boldsymbol{f}') \in \mathbb{R}\times\mathbb{R}^{N-1} : \boldsymbol{\psi}_{\boldsymbol{\sigma}'}^{(-)\prime}(\boldsymbol{f}^{(-)\prime})^{\mathrm{T}}\mathbf{1} + (-)_N\psi_{\sigma_N}((-)_N(\alpha - \boldsymbol{f}'^{\mathrm{T}}\mathbf{1})) = \beta,\ \ \beta \in B_{x,N}\right\}\right\}\right\}, \tag{25}$$

and then the following inclusion relationship holds:

$$\mathrm{supp}\left\{S_{\boldsymbol{y}}^{\alpha}(\boldsymbol{f}')_N\right\} \subseteq \bigcup_{\boldsymbol{\sigma}\in\Omega^N} \bigcup_{\beta\in B_{x,N}} \left\{(\alpha, \boldsymbol{f}') \in \mathbb{R}\times\mathbb{R}^{N-1} : \boldsymbol{\psi}_{\boldsymbol{\sigma}'}^{(-)\prime}(\boldsymbol{f}^{(-)\prime})^{\mathrm{T}}\mathbf{1} + (-)_N\psi_{\sigma_N}((-)_N(\alpha - \boldsymbol{f}'^{\mathrm{T}}\mathbf{1})) = \beta\right\}. \tag{26}$$

Therefore, for the considered conjugation configuration, the output time-series $y(t)$ can exhibit Nth-order WSCS with cycle frequency α only if, for some $\boldsymbol{\sigma} \in \Omega^N$ and $\beta \in B_{x,N}$, the set

$$\left\{\boldsymbol{f}' \in \mathbb{R}^{N-1} : \boldsymbol{\psi}_{\boldsymbol{\sigma}'}^{(-)\prime}(\boldsymbol{f}^{(-)\prime})^{\mathrm{T}}\mathbf{1} + (-)_N\psi_{\sigma_N}((-)_N(\alpha - \boldsymbol{f}'^{\mathrm{T}}\mathbf{1})) = \beta\right\} \tag{27}$$

has nonzero measure in $\mathbb{R}^{N-1}$ or has nonempty intersection with the β-submanifold [3] of $S_{\boldsymbol{x}}^{\beta}\left(\boldsymbol{\psi}_{\boldsymbol{\sigma}'}^{(-)\prime}(\boldsymbol{f}^{(-)\prime})\right)_N$. In particular, if the input time-series $x(t)$ exhibits Nth-order wide-sense stationarity for the given conjugation configuration (i.e., $S_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}')_N \not\equiv 0$ only for $\beta = 0$), from (26) it follows that

$$\mathrm{supp}\left\{S_{\boldsymbol{y}}^{\alpha}(\boldsymbol{f}')_N\right\} \subseteq \bigcup_{\boldsymbol{\sigma}\in\Omega^N} \left\{(\alpha, \boldsymbol{f}') \in \mathbb{R}\times\mathbb{R}^{N-1} :\right.$$

$$\psi_{\sigma'}^{(-)'}(f^{(-)'})^{\mathrm{T}}\mathbf{1} + (-)_N \psi_{\sigma_N}((-)_N(\alpha - f'^{\mathrm{T}}\mathbf{1})) = 0\Big\}. \tag{28}$$

In such a case, the output time-series $y(t)$ can exhibit Nth-order WSCS with cycle frequency α only if, for at least one value of $\sigma \in \Omega^N$, the set

$$\Big\{f' \in \mathrm{R}^{N-1} \ : \ \psi_{\sigma'}^{(-)'}(f^{(-)'})^{\mathrm{T}}\mathbf{1} + (-)_N \psi_{\sigma_N}((-)_N(\alpha - f'^{\mathrm{T}}\mathbf{1})) = 0\Big\} \tag{29}$$

has nonzero measure in R^{N-1} or has nonempty intersection with the 0-submanifold of $S_{x}^{0}\left(\psi_{\sigma'}^{(-)'}(f^{(-)'})\right)_N$.

The wide-sense almost-cyclostationary time-series are closed under LAPTV transformations. In fact, when the input time-series $x(t)$ is wide-sense almost-cyclostationary and, hence,

$$\mathrm{supp}\ \Big\{S_{x}^{\beta}(f')_N\Big\} \subseteq \Big\{(\beta, f') \in \mathrm{R}\times\mathrm{R}^{N-1} \ : \ \beta \in B_{x,N}\Big\}, \tag{30}$$

accounting for (7), it results that

$$\mathrm{supp}\ \Big\{S_{y}^{\alpha}(f')_N\Big\} \subseteq \bigcup_{\sigma\in\Omega^N}\ \bigcup_{\beta\in B_{x,N}} \Big\{(\alpha, f') \in \mathrm{R}\times\mathrm{R}^{N-1} \ : \ \alpha = \beta + \sigma^{(-)\mathrm{T}}\mathbf{1}\Big\}, \tag{31}$$

where $\sigma^{(-)} \triangleq [(-)_1\sigma_1, ..., (-)_N\sigma_N]^{\mathrm{T}}$. In addition, the wide-sense almost-cyclostationary time-series are closed under time scale changing. In fact, accounting for (10), one has

$$\mathrm{supp}\ \Big\{S_{y}^{\alpha}(f')_N\Big\} \subseteq \bigcup_{\beta\in B_{x,N}} \Big\{(\alpha, f') \in \mathrm{R}\times\mathrm{R}^{N-1} \ : \ \alpha = a\beta\Big\}. \tag{32}$$

When the considered linear time-variant system is strictly bandlimited in f (and possibly in λ), that is,

$$H(f,\lambda) = 0, \quad |f| \geq B/2, \quad \forall\lambda, \tag{33}$$

the following inclusion relation for the support of $S_{y}^{\alpha}(f')_N$ in the (α, f') space holds:

$$\begin{aligned}
&\mathrm{supp}\ \Big\{S_{y}^{\alpha}(f')_N\Big\} \subseteq \\
&\bigcup_{\sigma\in\Omega^N} \mathrm{supp}\left\{H_{\sigma_N}^{(*)_N}((-)_N(\alpha - f'^{\mathrm{T}}\mathbf{1}))\prod_{n=1}^{N-1} H_{\sigma_n}^{(*)_n}((-)_n f_n)\right\} \\
&\subseteq \mathrm{supp}\left\{\mathrm{rect}\left(\frac{\alpha - f'^{\mathrm{T}}\mathbf{1}}{B}\right)\prod_{n=1}^{N-1}\mathrm{rect}\left(\frac{f_n}{B}\right)\right\} \\
&= \Big\{(\alpha, f') \in \mathrm{R}\times\mathrm{R}^{N-1} \ : \ |\alpha - f'^{\mathrm{T}}\mathbf{1}| \leq B/2, |f_n| \leq B/2\Big\} \\
&\subseteq \Big\{(\alpha, f') \in \mathrm{R}\times\mathrm{R}^{N-1} \ : \ |\alpha| \leq NB/2, |f_n| \leq B/2\Big\}.
\end{aligned} \tag{34}$$

To derive (34), it has been assumed without loss of generality that for any $\sigma_1 \neq \sigma_2$ it results that $\varphi_{\sigma_1}(\lambda) = \varphi_{\sigma_2}(\lambda)$ at most in a denumerable set of values of λ and then, from (33), it follows that

$$H_{\sigma}(f) = 0, \quad |f| \geq B/2, \quad \forall\sigma. \tag{35}$$

Moreover, the last inclusion in (34) follows from

$$|\alpha - f'^{\mathrm{T}}\mathbf{1}| \geq |\alpha| - |f'^{\mathrm{T}}\mathbf{1}| \geq |\alpha| - (N-1)B/2, \tag{36}$$

which holds for $|f_n| \leq B/2$ $(n = 1, ..., N-1)$.

Equation (34) shows that the output time series $y(t)$ cannot exhibit Nth-order WSCS with cycle frequencies α such that $|\alpha| > NB/2$ independently of the WSCS properties of the input time-series.

In regard to the Nth-order cyclic temporal cross-moment function (CTCMF) of the M output time-series $y(t+\tau_n)$ $(n = 1, ..., M)$ and the $N-M$ input time-series $x(t+\tau_n)$ $(n = M+1, ..., N)$ defined as

$$\mathcal{R}_{y^{(M)}x^{(N-M)}}^{\alpha}(\tau)_N \triangleq \left\langle \prod_{n=1}^{M} y^{(*)_n}(t+\tau_n) \prod_{n=M+1}^{N} x^{(*)_n}(t+\tau_n)e^{-j2\pi\alpha t}\right\rangle, \tag{37}$$

accounting for (5), one has

$$\begin{aligned}
\mathcal{R}_{y^{(M)}x^{(N-M)}}^{\alpha}(\tau)_N = &\int_{\mathrm{R}^M}\int_{\mathrm{R}^M} \sum_{\sigma\in\Omega^M}\left(\prod_{n=1}^{M} h_{\sigma_n}^{(*)_n}(u_n)\right) \\
&\cdot \sum_{\beta\in B_{x,N}} \mathcal{R}_{x}^{\beta}([v^{(M)\mathrm{T}}, \tau^{(N-M)\mathrm{T}}]^{\mathrm{T}})_N \\
&\cdot \mathcal{R}_{\Psi_{\sigma_1}^{(*)_1},...,\Psi_{\sigma_M}^{(*)_M}}^{\alpha-\beta}(\tau^{(M)} - u^{(M)}, v^{(M)})_M \, \mathrm{d}u^{(M)}\, \mathrm{d}v^{(M)},
\end{aligned} \tag{38}$$

where the superscript (K) is the operator that transforms a vector $w \triangleq [w_1, ..., w_N]^{\mathrm{T}}$ into $w^{(K)} \triangleq [w_1, ..., w_K]^{\mathrm{T}}$ with $K < N$.

The reduced-dimension CTCMF $R_{y^{(M)}x^{(N-M)}}^{\alpha}(\tau')_N$ is obtained setting $\tau_N = 0$ in $\mathcal{R}_{y^{(M)}x^{(N-M)}}^{\alpha}(\tau)_N$. Moreover, by Fourier transforming both sides of (38) and accounting for (13) and (22), one obtains the input/output relation in terms of reduced-dimension cyclic spectral cross-moment functions.

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