OPTIMAL SUBARRAY DIMENSION FOR SPATIAL SMOOTHING TECHNIQUE

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ABSTRACT

The popular spatial smoothing technique is considered. We show via the covariance matrix eigenvalue analysis the existence of simple suboptimal rule for choosing of the subarray dimension in a practically important situation of two coherent equipower closely spaced sources. This rule has been found by maximizing the distance between the signal subspace and the noise subspace eigenvalues of spatially smoothed covariance matrix and it does not require any *a priori* information about the signal source parameters.

1. INTRODUCTION

The spatial coherence between signal sources is known to considerably decrease the performance of majority of high-resolution direction finding methods. The well known spatial smoothing preprocessing technique [1], [2], allowing one to remove the signal coherence and to improve the performance of high-resolution methods in the coherent environment, partitions the total uniform linear array (ULA) of n sensors into k overlapped subarrays with the dimension $m = n - k + 1$ and then generates the average of subarray output covariance matrices. In the case of the fixed number of sensors of a total array the dimension of a subarray can be considered as a free parameter of the spatial smoothing technique and, if the number of signal sources is small, this parameter is usually chosen from a compromise between the potential spatial resolution and the decorrelation effect. Actually, if the subarray dimension is small (i.e, the number of subarrays is large), a very good decorrelation can be achieved, while the potential spatial resolution becomes poor due to the small size of working aperture. In the opposite case, when the subarray dimension is large (i.e., number of subarrays is small), one can achieve a high potential resolution but the decorrelation effect may be unsatisfactory. Thus, the optimal subarray dimension exists, providing the best effectiveness of the spatial smoothing technique for a given dimension of the total array. Generally, the optimal subarray size depends on the source coordinates and on the relative phase between the sources which cannot be known *a priori*. But in some practically important asymptotic cases the optimal subarray size can be independent of the source parameters. In this paper (see also [3]) we show this fact and derive the optimal dimension of subarray for the asymptotic case of two coherent equipower closely spaced sources using the expressions for covariance matrix eigenvalues [4], [5].

2. COVARIANCE MATRIX EIGENVALUES

Consider a ULA of n sensors. The $n \times n$ covariance matrix of array outputs can be expressed as

$$\boldsymbol{R} = \boldsymbol{A}\boldsymbol{S}\boldsymbol{A}^H + \sigma^2 \boldsymbol{I} \tag{1}$$

where $\boldsymbol{I}$ is the identity matrix, σ^2 is the noise variance, H denotes the Hermitian transpose,

$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_q] \tag{2}$$

is the $n \times q$ matrix of wavefront vectors of q $(q < n)$ signal sources, $\boldsymbol{S}$ is the $q \times q$ covariance matrix of signal waveforms. The ith signal wavefront is given by a $n \times 1$ vector

$$\boldsymbol{a}_i = (e^{j\frac{2\pi}{\lambda}d(1-(n+1)/2)\sin\theta_i}, e^{j\frac{2\pi}{\lambda}d(2-(n+1)/2)\sin\theta_i}, \ldots, e^{j\frac{2\pi}{\lambda}d(n-(n+1)/2)\sin\theta_i})^T \tag{3}$$

where λ is the wavelength, d denotes the interelement spacing and T denotes the transpose.

The elements of the covariance matrix $\boldsymbol{S}$ are given by:

$$S_{il} = \sigma_i \sigma_l \rho_{il}, \qquad i, l = 1, 2, \ldots, q \tag{4}$$

where $\rho_{il} = |\rho_{il}| \exp\{j\phi_{il}\}$ is the complex coefficient of mutual correlation between ith and lth source and σ_i^2 is the variance of the waveform of the ith source.

The $m \times m$ spatially smoothed covariance matrix can be expressed as [2]:

$$\tilde{\boldsymbol{R}} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{S}}\tilde{\boldsymbol{A}}^H + \sigma^2 \boldsymbol{I} \tag{5}$$

Here $\tilde{\boldsymbol{A}}$ is the $m \times q$ matrix of subarray wavefront vectors:

$$\tilde{\boldsymbol{A}} = [\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2, \ldots, \tilde{\boldsymbol{a}}_q] \tag{6}$$

and the $m \times 1$ subarray vector of the ith wavefront differs from (3) only by its reduced dimension:

$$\tilde{\boldsymbol{a}}_i = (e^{j\frac{2\pi}{\lambda}d(1-(m+1)/2)\sin\theta_i}, e^{j\frac{2\pi}{\lambda}d(2-(m+1)/2)\sin\theta_i}, \ldots, e^{j\frac{2\pi}{\lambda}d(m-(m+1)/2)\sin\theta_i})^T \tag{7}$$

The elements of the $q \times q$ spatially smoothed covariance matrix of signal waveforms $\tilde{\boldsymbol{S}}$ are given by:

$$\tilde{S}_{il} = \sigma_i \sigma_l \tilde{\rho}_{il}, \qquad i, l = 1, 2, \ldots, q \tag{8}$$

where $\tilde{\rho}_{il} = |\tilde{\rho}_{il}| \exp\{j\psi_{il}\}$ is the complex coefficient of mutual correlation between ith and lth source after the spatial smoothing. It is easy to show that (see [2], [4], for example):

$$\tilde{\rho}_{il} = \rho_{il} g_{il}(k) e^{j(\pi d/\lambda)(k-1)(\sin\theta_i - \sin\theta_l)} \tag{9}$$

where

$$g_{il}(k) = \frac{\sin[k(\pi d/\lambda)(\sin\theta_i - \sin\theta_l)]}{k \sin[(\pi d/\lambda)(\sin\theta_i - \sin\theta_l)]} \tag{10}$$

Let $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$ be the eigenvalues of the covariance matrix (1). From (1) the well known fact follows that the noise subspace eigenvalues coincide with the noise variance [1], [2]:

$$\lambda_i = \sigma^2, \qquad i = q+1, q+2, \ldots, n \tag{11}$$

Consider now the case of two arbitrary correlated sources ($q = 2$). It was shown in [4], [5] that in this case the exact explicit expressions for signal subspace eigenvalues of the covariance matrix (1) can be written as

$$\lambda_{1,2} = \sigma^2 + n(\sigma_1^2 + \sigma_2^2)/2 + n\sigma_1\sigma_2 g(n)|\rho|\cos\phi \pm \left[\left(n(\sigma_1^2 + \sigma_2^2)/2 + n\sigma_1\sigma_2 g(n)|\rho|\cos\phi\right)^2 - n^2\sigma_1^2\sigma_2^2(1 - g^2(n))(1 - |\rho|^2)\right]^{\frac{1}{2}} \tag{12}$$

where for simplicity $g_{12}(n)$, ρ_{12}, and ϕ_{12} hereafter are rewritten as $g(n)$, ρ, and ϕ, respectively. If $\rho = 0$, the expressions (12) coincide with the well known expressions for uncorrelated sources [6]. If the sources are fully coherent, i.e., if $|\rho| = 1$, the second eigenvalue becomes equal to the noise subspace eigenvalue σ^2 because the rank of matrix $\boldsymbol{S}$ becomes equal to unity.

Let $\{\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_m\}$ be the eigenvalues of the spatially smoothed covariance matrix (5). From (5) it follows that the noise subspace eigenvalues

$$\tilde{\lambda}_i = \sigma^2, \qquad i = q+1, q+2, \ldots, m \tag{13}$$

The comparison of the conventional and spatially smoothed covariance matrices (1) and (5) shows that they have the same structure. Namely, they differ only by dimension and by correlation coefficients in the signal waveform covariance matrices (4) and (8). The ilth element of matrix (4) contains ρ_{il}, while the ilth element of matrix (8) contains $\tilde{\rho}_{il}$, where the relationship between ρ_{il} and $\tilde{\rho}_{il}$ is given by (9). Using this property in the case of two sources, we can obtain the signal subspace eigenvalues of spatially smoothed covariance matrix from signal subspace eigenvalues (12) by replacing $n \to m$, $\rho \to \tilde{\rho}$, where hereafter $\tilde{\rho}_{12}$ is rewritten as $\tilde{\rho} = |\tilde{\rho}| \exp\{j\psi\}$. Therefore, taking into account (9), we can express the signal subspace eigenvalues of the spatially smoothed covariance matrix as:

$$\tilde{\lambda}_{1,2} = \sigma^2 + m(\sigma_1^2 + \sigma_2^2)/2 + m\sigma_1\sigma_2 g(m)g(k)|\rho|\cos\psi \pm \left[\left(m(\sigma_1^2 + \sigma_2^2)/2 + m\sigma_1\sigma_2 g(m)g(k)|\rho|\cos\psi\right)^2 - m^2\sigma_1^2\sigma_2^2(1 - g^2(m))(1 - g^2(k)|\rho|^2)\right]^{\frac{1}{2}} \tag{14}$$

where

$$\psi = \phi + (\pi d/\lambda)(k-1)(\sin\theta_1 - \sin\theta_2) \tag{15}$$

3. OPTIMAL SUBARRAY SIZE

Let us derive the optimal rule for choosing the subarray size for the special case of two fully coherent equipower closely spaced sources. It is well known (see also (12)) that the mutual coherence between the sources reduces the distance between the signal subspace and the noise subspace eigenvalues. The goal of the spatial smoothing technique is to increase this distance. Therefore, the optimal subarray dimension for the fixed number of the total array sensors can be determined by maximizing the eigenvalue distance $D = \tilde{\lambda}_q - \tilde{\lambda}_{q+1}$. Such a criterion is also correct in the finite sample case, because the finite sample eigenvalue distance can be considered as a small $O(1/\sqrt{N})$ perturbation of the exact eigenvalue distance when the number of samples N is large enough. Assuming $|\rho| = 1$, $\sigma_1^2 = \sigma_2^2 = \sigma_S^2$, we get from (14):

$$D = \tilde{\lambda}_2 - \sigma^2 = m\sigma_S^2 \Bigg\{ (1 + g(m)g(k)\cos\psi) - \left[(1 + g(m)g(k)\cos\psi)^2 - (1 - g^2(m))(1 - g^2(k)) \right]^{\frac{1}{2}} \Bigg\} \quad (16)$$

Applying the expansion $\sin x = x - x^3/3! + \cdots$ to (10) and assuming that n is large enough (and, therefore, k is also large), and that the sources are closely spaced, after neglecting the small high-order terms we have:

$$g_{il}(k) \simeq 1 - \frac{k^2}{6}\left(\frac{\pi d}{\lambda}\right)^2 (\sin\theta_1 - \sin\theta_2)^2 \quad (17)$$

Substitution of (17) yields that for all the values of $\cos\psi$, excepting the values which are very close to -1

$$\frac{(1 - g^2(k))(1 - g^2(m))}{(1 + g(m)g(k)\cos\psi)^2} = O\left((\sin\theta_1 - \sin\theta_2)^4\right) \ll 1 \quad (18)$$

because the sources are assumed to be closely spaced. Equation (18) enables to apply the expansion $\sqrt{1-x} = 1 - (1/2)x - \cdots$ to (16), and, using (17), we find after the neglecting the high-order terms that

$$D \simeq \frac{m\sigma_S^2}{2} \frac{(1 - g^2(m))(1 - g^2(k))}{1 + g(m)g(k)\cos\psi} \simeq \frac{\sigma_S^2}{18} \frac{m^3 k^2 ((\pi d/\lambda)(\sin\theta_1 - \sin\theta_2))^4}{1 + \cos\psi} \quad (19)$$

where we assume once more that $\cos\psi$ is not very close to -1. Substituting $k = n - m + 1$ in (19) and taking into account that $\cos\psi$ also depends on m, we have that the optimal value of the parameter m can be found by maximizing the function

$$f(m) = \frac{m^3(n - m + 1)^2}{1 + \cos\psi} \quad (20)$$

with respect to m. Taking the first derivative with use of (15) we have

$$f'(m) = \frac{G_1 - G_2(\pi d/\lambda)\sin\psi}{(1 + \cos\psi)^2} \quad (21)$$

where

$$G_1 = (1 + \cos\psi)F'(m),$$
$$G_2 = (\sin\theta_1 - \sin\theta_2)F(m), \quad (22)$$
$$F(m) = m^3(n - m + 1)^2$$

From equations (22) it follows that $G_1 = O(m^4)$, while $G_2 = O(m^5(\sin\theta_1 - \sin\theta_2))$. Using once more the assumption of closely spaced sources, i.e., $m|\sin\theta_1 - \sin\theta_2| \ll 1$, we get that $G_2 \ll G_1$ and, therefore, the term with G_2 can be ignored in (21). It yields the optimal subarray size:

$$m_{opt} = 0.6(n + 1) \quad (23)$$

which is independent of the source coordinates, of the phase ψ and of the signal power σ_S^2.

4. NUMERICAL EXAMPLE

In order to verify the optimality of (23) we consider a simple numerical example. We assumed a ULA with half-wavelength spacing and two fully coherent closely spaced sources with angular separation corresponding to $g(n) = 0.972$. We calculated the optimal m using (23) (with rounding to the closest integer) and also straightforwardly, by exact maximization the eigenvalue distance (16) with respect to m for different dimensions of total array (n = 10, 25 and 50) and for different phase values ($\phi = 0$, $\pi/4$, $\pi/2$, and π). For $n = 10$ eqn.(23) yields $m_{opt} = 7$ while the straightforward calculations yield $m_{opt} = 6, 7, 7, 7$ for $\phi = 0$, $\pi/4$, $\pi/2$, and π, respectively. For $n = 25$ eqn.(23) yields $m_{opt} = 16$, while the straightforward maximization of the eigenvalue distance yields $m_{opt} = 16$, 16, 16, and 17 for $\phi = 0, \pi/4, \pi/2$, and π, respectively. At last, for $n = 50$ we get from (23) that $m_{opt} = 31$, while the straightforward computations yield $m_{opt} = 30$, 31, 31, and 34. We see that the values obtained from (23) are very close to the exact values for the optimal subarray dimension for various total array dimensions and relative phases.

5. CONCLUSION

We address the problem of choosing the optimal subarray dimension in the popular spatial smoothing method for the case of fixed total array dimension and two coherent equipower closely spaced sources. The criterion of the maximum eigenvalue distance between the signal and the noise subspaces have been utilized for optimization of the subarray size. It has been shown that the optimal subarray size is very simply related with the total array size and that it does not depend on the signal source parameters.

6. ACKNOWLEDGMENTS

The work of the first author was supported by Alexander von Humboldt Foundation under Research Fellowship and by INTAS under SASPARC project INTAS-93-642.

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