STOCHASTIC AND DETERMINISTIC CRAMER–RAO BOUNDS ON DIRECTION-OF-ARRIVAL ESTIMATION ACCURACY IN THE PRESENCE OF UNKNOWN NOISE FIELD

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ABSTRACT

In this paper, the stochastic and deterministic Cramer-Rao Bounds (CRB's) on Direction-Of-Arrival (DOA) estimation accuracy in the presence of uncorrelated unknown noise field are derived. The presented CRB's allow to provide significant insights into the case of unknown sensor noise. In particular, it is shown that sensors with relatively powerful noises have a weak influence on the potential DOA estimation accuracy.

1. INTRODUCTION

A powerful tool for the study of potential DOA estimation performance in sensor arrays is the CRB showing the lower bound of estimation errors [1], [2]. The importance of the CRB is due to the well known fact that several DOA estimators like MUSIC [1], MODE [2], and stochastic ML method [3], [4] attain CRB under various conditions [1], [2].

The general expressions for CRB on DOA estimation in a sensor array in the case of identical noise powers were derived in [1], [2] both for deterministic and stochastic signal models. However, the assumption of equal sensor noise powers is often unrealistic [5] because of various nonidealities of receiving channels [6]. Moreover, often in practice it is impossible to measure the noise powers in the absence of signal and the assumption of *a priori* known sensor noise powers is unrealistic too. In this paper we present the expressions for stochastic and deterministic CRB's on DOA estimation for a more realistic case of uncorrelated and unknown noise field. For simplicity, the single source scenario is considered. The analysis of CRB's behavior is presented, showing some peculiarities inherent in the case of nonequal sensor noise powers.

2. DATA MODELS

Consider a narrow-band sensor array of n sensors and of arbitrary geometry. Let single source impinges on the array from direction θ. Hence, the $n \times 1$ vector of complex array outputs can be modeled as [1], [2]:

$$\boldsymbol{x}(t) = s(t)\boldsymbol{a} + \boldsymbol{n}(t), \qquad t = 1, 2, \ldots, N \qquad (1)$$

where N is the number of array data snapshots available, $\boldsymbol{a}$ is the $n \times 1$ source steering vector, $s(t)$ is the source waveform, and $\boldsymbol{n}(t)$ is the $n \times 1$ vector of additive sensor noise. The noise is assumed to be stationary, statistically independent, and zero-mean complex Gaussian vector with the $n \times n$ diagonal spatial covariance matrix

$$\boldsymbol{Q} = \mathrm{E}\{\boldsymbol{n}(t)\boldsymbol{n}^H(t)\} = \mathrm{diag}\{p_1, p_2, \ldots, p_n\} \qquad (2)$$

where p_i is an unknown noise power in i-th sensor, H denotes Hermitian transpose.

In stochastic case the signal waveform is assumed to be stationary, zero-mean complex Gaussian quantity [2]. Therefore, in this case:

$$\boldsymbol{x}(t) \sim \mathcal{N}_n^C(0, \boldsymbol{R}) \qquad (3)$$

where

$$\boldsymbol{R} = \boldsymbol{Q} + p_S \boldsymbol{a}\boldsymbol{a}^H \qquad (4)$$

and $p_S = \mathrm{E}\{|s(t)|^2\}$.

In deterministic case the signal waveform is assumed to be a deterministic process, i.e., to remain fixed from realization to realization but to vary randomly from snapshot to snapshot [2], [7]. In other words, under deterministic assumption:

$$\boldsymbol{x}(t) \sim \mathcal{N}_n^C(s(t)\boldsymbol{a}, \boldsymbol{Q}) \qquad (5)$$

The data models (3) and (5) lead to different CRB's on DOA estimation.

3. DERIVATION OF CRAMER-RAO BOUNDS

It is well known that for any unbiased estimate $\hat{\boldsymbol{\alpha}}$ of vector parameter $\boldsymbol{\alpha}$ the CRB is given by the diagonal elements of the inverted Fisher information matrix and it shows the lower bound of parameter estimation errors, i.e.:

$$\mathrm{var}(\hat{\alpha}_i) \geq \mathrm{CRB}(\hat{\alpha}_i)$$

where "hat" sign denotes estimate.

For stochastic model the vector of unknown parameters can be represented as:

$$\boldsymbol{\alpha} = (\omega, p_S, p_1, \ldots, p_n)^T \tag{6}$$

while for deterministic model

$$\boldsymbol{\alpha} = (\omega, \mathrm{Re}(s(1)), \ldots, \mathrm{Re}(s(N)), \mathrm{Im}(s(1)), \ldots, \mathrm{Im}(s(N)), p_1, \ldots, p_n)^T \tag{7}$$

where $\omega = \sin\theta$ and T denotes transpose.

The use of the Bangs formula [2] together with (2)-(7) allows to calculate the $(n+2) \times (n+2)$ and the $(2N+n+1) \times (2N+n+1)$ Fisher information matrices corresponding to the stochastic and deterministic models, respectively. Fortunately, for both cases these matrices have a structure which enables to inverse them analytically and to find the explicit expressions for CRB. After lengthy derivations including such matrix inversion, we get the following expressions for CRB's on DOA estimation for stochastic and deterministic signal models, respectively:

$$\mathrm{CRB}_{\mathrm{ST}}(\hat{\omega}) = \{1 + p_S \boldsymbol{a}^H \boldsymbol{Q}^{-1} \boldsymbol{a}\} / \{2N p_S^2 [\boldsymbol{a}^H \boldsymbol{Q}^{-1} \boldsymbol{a} \boldsymbol{a}^H \boldsymbol{D} \boldsymbol{Q}^{-1} \boldsymbol{D} \boldsymbol{a} - (\boldsymbol{a}^H \boldsymbol{D} \boldsymbol{Q}^{-1} \boldsymbol{a})^2]\} \tag{8}$$

$$\mathrm{CRB}_{\mathrm{DET}}(\hat{\omega}) = \{\boldsymbol{a}^H \boldsymbol{Q}^{-1} \boldsymbol{a}\} / \{2N \tilde{p}_S [\boldsymbol{a}^H \boldsymbol{Q}^{-1} \boldsymbol{a} \boldsymbol{a}^H \boldsymbol{D} \boldsymbol{Q}^{-1} \boldsymbol{D} \boldsymbol{a} - (\boldsymbol{a}^H \boldsymbol{D} \boldsymbol{Q}^{-1} \boldsymbol{a})^2]\} \tag{9}$$

where the $n \times n$ diagonal matrix

$$\boldsymbol{D} = \frac{2\pi}{\lambda} \mathrm{diag}\{d_1, d_2, \ldots, d_n\} \tag{10}$$

the $n \times 1$ steering vector

$$\boldsymbol{a} = (|a_1| \exp\{j(2\pi/\lambda) d_1 \omega\}, \ldots \ldots, |a_n| \exp\{j(2\pi/\lambda) d_n \omega\})^T \tag{11}$$

λ is wavelength, d_i is the coordinate of i-th sensor, and

$$\tilde{p}_S = \frac{1}{N} \sum_{i=1}^{N} |s(i)|^2 \tag{12}$$

It is easy to verify that in the special case of identical noise powers and of uniform linear array the expressions (8) and (9) coincide with the simple familiar expressions (see, for example, [7]).

4. ANALYSIS AND COMPARISON OF CRAMER-RAO BOUNDS

Let us analyse the relationship between stochastic and deterministic CRB's via comparison (8) and (9). Assume that the number of snapshots is large, i.e., $N \gg 1$. In this case one can suppose asymptotically that $p_S = \tilde{p}_S$. Under this assumption, the deterministic CRB is always lower than the stochastic one and the difference between these two bounds becomes negligible when

$$p_S \boldsymbol{a}^H \boldsymbol{Q}^{-1} \boldsymbol{a} \gg 1 \tag{13}$$

Equation (13) is not more than the high Signal-to-Noise Ratio (SNR) condition. Therefore, in the case of nonidentical sensor noise powers the SNR in a whole array can be defined as

$$\mathrm{SNR} = p_S \boldsymbol{a}^H \boldsymbol{Q}^{-1} \boldsymbol{a} \tag{14}$$

Assuming for simplicity that $|a_i| = 1, \quad i = 1, 2, \ldots, n$, rewrite (14) as

$$\mathrm{SNR} = p_S \sum_{i=1}^{n} \frac{1}{p_i} = \sum_{i=1}^{n} \mathrm{SNR}_i$$

where $\mathrm{SNR}_i = p_S / p_i$ is SNR in the i-th sensor.

In the low SNR case (i.e., when $\mathrm{SNR} \ll 1$), the relationship between stochastic and deterministic CRB is given by:

$$\mathrm{CRB}_{\mathrm{DET}}(\hat{\omega}) = \mathrm{SNR}\, \mathrm{CRB}_{\mathrm{ST}}(\hat{\omega}) \tag{15}$$

Therefore, the ratio between the deterministic and stochastic CRB decreases as the SNR decreases. These results reconfirm the results of Stoica and Nehorai, obtained in [2] for a more simple case of identical noise powers but for arbitrary number of sources.

If the number of snapshots is small, $\tilde{p}_S$ and p_S may essentially differ and in some situations with relatively high SNR it can happen that the stochastic CRB is even lower that the deterministic one. It depends on the temporal behaviour of the deterministic process $s(t)$.

4.1 Case of distinguished sensors with powerful noises

One of the most interesting questions appearing when the case of nonidentical sensor noises is analysed, is the following: how depends the potential DOA estimation accuracy on the sensors with relatively powerful noises?

In order to answer this question, let us analyse the special case when several sensors have relatively powerful noises as compared with other sensors. In other words, let the noise powers in arbitrary K sensors ($K < n-1$) with the numbers $l_1, \dots, l_K$ are much higher than that in the other $n - K$ sensors, i.e.:

$$p_{l_m} \gg p_k, \qquad m = 1, \dots, K, \qquad k \neq l_1, \dots, l_k \quad (16)$$

Then, the matrix $\boldsymbol{Q}^{-1}$ can be divided in two components:

$$\boldsymbol{Q}^{-1} = \boldsymbol{P}_{[K]} + \boldsymbol{P}_{[n-K]} \quad (17)$$

where $\boldsymbol{P}_{[K]}$ is formed from the matrix $\boldsymbol{Q}^{-1}$ by replacing $n - K$ diagonal elements, corresponding to the sensors with relatively low noise powers, by zeros. In turn, $\boldsymbol{P}_{[n-K]}$ represents the matrix $\boldsymbol{Q}^{-1}$ after replacing the diagonal elements, corresponding to distinguished K sensors, by zeros. From (16) we have that $||\boldsymbol{P}_{[K]}|| \ll ||\boldsymbol{P}_{[n-K]}||$, and, therefore,

$$\boldsymbol{a}^H \boldsymbol{Q}^{-1} \boldsymbol{a} \simeq \boldsymbol{a}^H \boldsymbol{P}_{[n-K]} \boldsymbol{a} \quad (18)$$

$$\boldsymbol{a}^H \boldsymbol{D} \boldsymbol{Q}^{-1} \boldsymbol{a} \simeq \boldsymbol{a}^H \boldsymbol{D} \boldsymbol{P}_{[n-K]} \boldsymbol{a} \quad (19)$$

$$\boldsymbol{a}^H \boldsymbol{D} \boldsymbol{Q}^{-1} \boldsymbol{D} \boldsymbol{a} \simeq \boldsymbol{a}^H \boldsymbol{D} \boldsymbol{P}_{[n-K]} \boldsymbol{D} \boldsymbol{a} \quad (20)$$

From (18)-(20) it follows that the sensors with relatively powerful noises have asymptotically no influence on the stochastic and deterministic CRB's (8), (9) and on performance of asymptotically efficient DOA estimators. It shows that DOA estimators that are asymptotically efficient in the case considered, estimate the noise powers as well and take this information into account when estimating DOA's.

5. CONCLUSION

The stochastic and deterministic CRB's on DOA estimation accuracy in the presence of uncorrelated unknown noise field are derived. The consideration was limited by a single source scenario. The analysis of the bounds obtained shows that they are related in a very simple manner. It is shown that the sensors with relatively powerful noises have weak influence on the potential DOA estimation accuracy. This fact provide the additive insights into DOA estimation problem in the case of different sensor noises.

ACKNOWLEDGMENTS

This work was supported by Alexander von Humboldt Foundation under Research Fellowship of the first author and by INTAS under SASPARC project INTAS-93-642.

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