Spatio-Temporal Equalizability under Channel Noise and Loss of Disparity

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Résumé

La plupart des algorithmes adaptatifs pour l'égalisation aveugle ont été proposés et étudiés en l'absence de bruit. Profitant de récents résultats analytiques sur l'égalisation spatio-temporelle, nous étudions l'effet du bruit de canal sur les performances de l'égalisation. L'égalisabilité du canal en présence de bruit est quantifiée en termes de puissance d'erreur entrée / sortie. Ceci fournit une borne minimale d'erreur à partir de laquelle une comparaison de la robustesse des performances d'algorithmes adaptatifs peut être effectuée. En particulier, le compromis réalisé par l'algorithme de Godard entre l'égalisation parfaite et l'amplification de la puissance du bruit sera mis en évidence.

Abstract

Most adaptive blind equalization algorithms have been proposed and studied in the noise-free context. Recent analitycal results on fractionally spaced equalization allow to study the effect of channel noise on equalization performances. Channel equalizability in noisy context is measured by the input / output error power. The mean square error gives a lower bound to which the performances of adaptive algorithms are compared. More specifically, our study shows how the Constant Modulus Algorithm (CMA) compromises between perfect equalization and noise enhancement.

1. INTRODUCTION

The spatio-temporal channel equalization problem consists on choosing the $L \times 1$ Finite Impulse Response (FIR) equalizer transfer function $(e_1(z), ..., e_L(z))^\top$ so that its output $y(n)$ acheives a "good" estimate of the delayed input sequence $s(n)$, as displayed on Figure 1, where $(c_1(z), ..., c_L(z))^\top$ represents the L-dimensional channel transfer function and $(w_1(n), ..., w_L(n))^\top$ the L-dimensional channel noise.

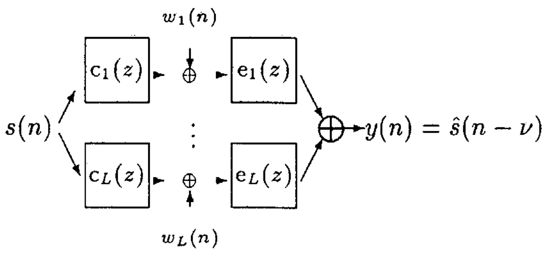

Figure 1: Spatio-Temporal Equalization Scheme

The multidimensional channel model derives from either temporal or spatial diversity (see [1] for example). Each of its components is assumed to be FIR with degree less or equal to Q. Recent studies (see [2], [3] for example) show that under the following conditions,

1. $c_1(z), ..., c_L(z)$ share no common zero,
2. each $e_i(z)$ $(i = 1, ..., L)$ is FIR with degree $N - 1 \geq Q$,
3. $s(n)$ is i.i.d.,
4. no channel noise.

perfect equalization is acheivable. More precisely, any combined channel / equalizer of $(Q+N)$-length impulse response is acheivable. Several adaptive blind (i.e., without knowledge of the input sequence) equalization algorithms have been derived from this property, [4] or [5] for example. These algorithms are strongly dependant on conditions 1-4. An important question is how their performances, and especially their ability to equalize, are affected when the previous conditions are no longer met. The relaxation of condition 1 is analytically evaluated in [6] in terms of input / output error power. The previous algorithms fail but Fractionally Spaced Equalizer Constant Modulus Algorithm (FSE-CMA, see [1]) performances show some robustness to this condition. The relaxation of condition 2 is addressed in [7]. In this paper, our purpose is to study the effect of suppressing condition 4, while maintening or not condition 1.

In order to understand the effect of channel noise, we first introduce the concept of **channel equalizability**, i.e. the ability of the channel to be equalized. Equalizability may be measured by the value of the input / output Minimum Mean Square Error (MMSE), $\min_{e,\nu} E[(y(n) - s(n-\nu))^2]$, for a given class of equalizers (here, LN-long FIR filter). The previously defined MMSE provides a lower bound in the error power within the class of linear NL-long equalizers.

Organization: In the second section, we recall results on the channel convolution matrix properties. Equalizability is introduced in Section 3. In the noisy context, the best achievable equalizers settings are estimated and the corresponding MMSE between the delayed input and output signal is deduced. In section 4, the effect of noise on the steady-state input / output MMSE of FSE-CMA is compared to the MMSE lower bound.

2. CHANNEL CONVOLUTION MATRIX

In this section, we set some notation and recall a few properties on the channel represented by its convolution matrix.

From the propagation model of Figure 1, one can represent the global (channel + equalizer) transfer function as

$$\mathrm{h}(z) = \mathrm{c}_1(z)\mathrm{e}_1(z) + \cdots + \mathrm{c}_L(z)\mathrm{e}_L(z) \qquad (1)$$

The impulse response h associated to equation (1) is:

$$h = \mathrm{C}^\top \vec{e} \qquad (2)$$

where the entries of h are the coefficients of $\mathrm{h}(z)$, and the entries of $\vec{e}$ are the coefficients of $(\mathrm{e}_1(z), \cdots, \mathrm{e}_L(z))^\top$. C is the $NL \times (Q+N)$ channel **convolution matrix** defined by the coefficients of $\vec{c}(z) = (\mathrm{c}_1(z), ..., \mathrm{c}_L(z))^\top$ as:

$$\mathrm{C} = \begin{bmatrix} c_1(0) & c_1(1) & .. & c_1(Q) & 0 & & \dots & 0 \\ \vdots & & & & & & & \\ c_L(0) & c_L(1) & .. & c_L(Q) & 0 & & \dots & 0 \\ 0 & c_1(0) & c_1(1) & .. & c_1(Q) & 0 & \dots & 0 \\ 0 & c_L(0) & c_L(1) & .. & c_L(Q) & 0 & \dots & 0 \\ & & & \ddots & & & & \\ 0 & .. & & 0 & c_1(0) & c_1(1) & \dots & c_1(Q) \\ 0 & .. & & 0 & c_L(0) & c_L(1) & \dots & c_L(Q) \end{bmatrix}$$

The range of $\mathrm{C}^\top$ determines achievable channel-equalizer impulse responses in the noise-free context.

Under conditions 1-2, C has been proved to be full column-rank, see [8], so that any $(N+Q)$-length impulse response h is achievable, in particular canonical vectors, denoted h_ν. Furthermore, as soon as $N \geq Q$, there exist a $(NL-(N+Q))$-dimensional subspace of seetings of $\vec{e}$ corresponding to the same value of h. This non-uniqueness may lead to numerical problems ([9]) when updating adaptive algorithms.

Under condition 1, with all $\mathrm{c}_k(z)$ having Z_0 common zeros, C has a rank equal to $(Q - Z_0 + N)$ ([10]). C admits a factorization as $\mathrm{C} = \underline{\mathrm{C}}\,\mathrm{C}_0$ where C_0 is the $(Q-Z_0+N) \times (Q+N)$ convolution matrix associated to the scalar polynomial corresponding to the Z_0 common zeros $(\mathrm{c}_0(z))$, and where $\underline{\mathrm{C}}$ is the $(NL) \times (Q - Z_0 + N)$ convolution matrix associated to the remaining part of the multichannel transfer function. Note that $\underline{\mathrm{C}}$ is full column-rank, but C_0 is full row-rank. The equalizability under condition 1, 3 and 4 is also equivalent to that of the scalar channel transfer function $\mathrm{c}_0(z)$, see [6].

Our main concern in the noisy channel context is noise enhancement since the additive noise is filtered by the equalizer only, the norm of which is inversely proportional to the difference between subchannels zeros under conditions 1-2. We will show in the following that equalizers can compromise between achieving a value of h close to the noise-free ideal h_ν and a small norm of $\vec{e}$.

3. EQUALIZABILITY

In this section, we express the best achievable equalizer in terms of Minimum Mean Square Error (MMSE) between the equalizer output and a delayed input sequence.

From the independance between the source signal and noise, the input / output mean square error writes as

$$\begin{aligned} f(\vec{e}) &= E[(y(n) - s(n-\nu))^2]/E[s^2(n)] \\ &= \|\mathrm{C}^\top \vec{e} - (0...010...0)^\top\|^2 + \vec{e}^\top \frac{\mathcal{R}_w}{E[s^2(n)]}\vec{e} \qquad (3) \end{aligned}$$

where $\|.\|$ denotes the euclidian norm and $E[.]$ stands for the mean expectation operator. $\mathcal{R}_w$ is the covariance matrix of the noise vector $(w_1(n), .., w_1(n-N+1), ..., w_L(n), .., w_L(n-N+1))^\top$. The minimization of (3) can thus be viewed as the minimization of the noiseless cost-function under a constraint on the equalizer norm in the noise covariance matrix sense. $\vec{e}^\top \frac{\mathcal{R}_w}{E[s^2(n)]}\vec{e}$, a weighted squared-norm of $\vec{e}$, can be viewed as a smoothing factor of the cost-function.

The MMSE is achieved for $\vec{e}$ satisfying

$$\left(\mathrm{C}\mathrm{C}^\top + \frac{1}{E[s^2(n)]}\mathcal{R}_w\right)\vec{e} = \mathrm{C}h_\nu \qquad (4)$$

where $h_\nu = (0...010...0)^\top$ is a $(N+Q)$-long vector with all components but the $(\nu+1)^{\text{th}}$ set to zero. Furthermore, in the contrary of the noise free case, R_w positive-definiteness increases the condition number of the right-handside matrix in (4) proportionally to the noise to signal power. This yields to the unicity of the solution and takes care of the numerical problems in the noise-free case, as soon as the noise power is large enough.

A first question is then when does equation (4) admit solutions ? Then, one may want to know the remaining MMSE, and the quantity of noise enhancement.

<u>Under conditions 1, 2, 3:</u>

In the noisy case and under conditions 1-2, $\mathrm{C}^\top\mathrm{C}$ is full rank. Therefore, (4) is equivalent to

$$\left(\mathrm{C}^\top + (\mathrm{C}^\top\mathrm{C})^{-1}\mathrm{C}^\top \frac{1}{E[s^2(n)]}R_w\right)\vec{e} = h_\nu \qquad (5)$$

The proof of this result and of the ones below involves only the matrix inversion lemma and some straightforward algebra, they are omitted. The left-handside matrix still being full row-rank, the equation admits solutions.

The noise to input power is defined by $\gamma = \mathrm{trace}(R_w)/(NLE[s^2])$, so that when the noise is temporally and spatially white $R_w = \gamma I$. In the white noise case and for small enough γ, the global channel equalizer setting h that corresponds to a solution of (4) admits an approximation in terms of γ as

$$h = h_\nu - \gamma(\mathrm{C}^\top\mathrm{C})^{-1}h_\nu + o(\gamma)$$

This expression shows that noise introduces some Inter-Symbol Interference (ISI) with respect to the optimal noise-free h_ν. The MMSE induced by these impulse responses, $f(\vec{e}) = \|h - h_\nu\|^2 + \gamma\|\vec{e}\|^2$ can also be approximated from (3) by

$$f(\vec{e}) = \gamma\|\vec{e}_\nu\|^2 + o(\gamma)$$

where $\vec{e}_\nu$ is the 0 order approximation (with respect to γ) of the solution of (4). Since there is a $(N(L-1)-Q)$-dimensional subspace of settings corresponding to each value of h, the one of interest value here is the minimal norm setting amoung the subspace corresponding to h, namely $\vec{e}_\nu = \mathrm{C}(\mathrm{C}^\top\mathrm{C})^{-1}h$. Note that this setting is the projection of any $\vec{e}$ such as $h = \mathrm{C}^\top\vec{e}$ on the subspace spanned by the columns of C. A first order approximation of $\vec{e}_\nu$, now unique solution of (4), is given by $\vec{e}_\nu = \mathrm{C}(\mathrm{C}^\top\mathrm{C})^{-1}h_\nu - \gamma\mathrm{C}(\mathrm{C}^\top\mathrm{C})^{-2}h_\nu + o(\gamma)$. Thus,

$$f(\vec{e}) = \gamma h_\nu^\top(\mathrm{C}^\top\mathrm{C})^{-1}h_\nu + o(\gamma) \qquad (6)$$

This indicates that the delay of the combined channel-equalizer achieving the MMSE is the one minimizing the

Rayleigh ratio $h_\nu^T(C^T C)^{-1}h_\nu$ over $\nu = 1, ..., N+Q$. Unfortunatly, it is of course impossible to constraint the delay to the optimal one.

Note that the output noise to output useful signal power ratio, $\gamma\|\vec{e}\|^2/\|h\|^2$ has the same first order approximation (with respect to γ) as the MMSE (6).

Next, we simulate the best achievable MMSE and corresponding channel-equalizer impulse response h versus SNR (SNR= $10Log_{10}\gamma$ dB). These simulations were computed in 2-dimensional chanel model where each transfer function is discribed by its zeros. The zeros of $c_1(z)$ are -1.4 and 0.6, the zeros of $c_2(z)$ are -0.4 and 1.1. In Figure 2, we can see that, even for a small SNR, the taps of h are very close to the noise-free value h_ν. In Figure 3, we want to validate expression (6). We compare the theorical MMSE (exactly calculated from (5)) to the approximation in (6). The simulation displays the accuarcy of the approximation even for a small SNR.

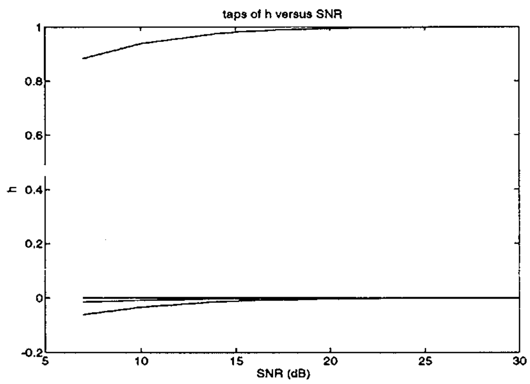

Figure 2: h versus SNR

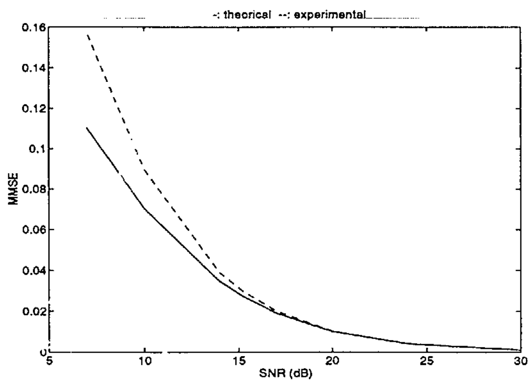

Figure 3: MMSE versus SNR

Under conditions 2, 3:

If condition 1 is suppressed, the noiseless cost-function may no longer be reduced to zero ([1]), since the range of C^T is a $(N+Q-Z_0)$-dimensional subspace in which none of the h_ν may lay. From the factorization of the convolution matrix $C = \underline{C}C_0$, with $C_0C_0^T$ and $\underline{C}^T\underline{C}$ being full rank, the solutions of (4) satisfy

$$\left(\underline{C}^T + (C_0C_0^T)^{-1}(\underline{C}^T\underline{C})^{-1}\underline{C}^T\frac{1}{E[s^2(n)]}R_w\right)\vec{e}$$
$$= (C_0C_0^T)^{-1}C_0h_\nu$$

Since $\underline{C}$ is full-column rank, there exist a solution to this equation the uniqueness of which is determined by the noise to signal power as in the previous case. For a white noise, the new setting minimizing the MMSE admits an approximation in terms of γ as

$$\begin{aligned} h &= C_0^T(C_0C_0^T)^{-1}C_0h_\nu \\ - &\quad \gamma C_0^T(C_0C_0^T)^{-1}(\underline{C}^T\underline{C})^{-1}(C_0C_0^T)^{-1}C_0h_\nu + o(\gamma) \end{aligned} \quad (7)$$

Note that $\Pi_0 = C_0^T(C_0C_0^T)^{-1}C_0$ is the orthogonal projector on the range of C_0, so that the non-avoidable error (in noise-free conditions) $\|(I-\Pi_0)h_\nu\|^2$ corresponds to the distance between h_ν and the range of C_0.

Note that $\|h - h_\nu\|^2 = \|(I-\Pi_0)h_\nu\|^2 + o(\gamma)$. Moreover, since the minimal norm $\vec{e}$ corresponding to a given h is equal to $\vec{e} = \underline{C}(\underline{C}^T\underline{C})^{-1}(C_0C_0^T)^{-1}C_0h$, the MMSE corresponding to (7) can be approximated by

$$\begin{aligned} f(\vec{e}) &= \|(I-\Pi_0)h_\nu\|^2 \\ + &\quad \gamma h_\nu^T C_0^T(C_0C_0^T)^{-1}(\underline{C}^T\underline{C})^{-1}(C_0C_0^T)^{-1}C_0h_\nu + o(\gamma) \end{aligned}$$

Thus, the MMSE corresponds to the delay ν compromizing between the minimization of $\|(I-\Pi_0)h_\nu\|^2$ (or in other terms maximization of $h_\nu^T\Pi_0h_\nu$) and the minimization of $h_\nu^T\Lambda_0^T(\underline{C}^T\underline{C})^{-1}\Lambda_0h_\nu$ weighted by γ, where $\Lambda_0 = (C_0C_0^T)^{-1}C_0$ denotes a generalized pseudoinverse of C_0. Of course, since one doesn't know how to constraint the delay, the lower norm $\vec{e}$ may not be reached, so that the noice enhancement may be much greater.

The main concern here is noise enhancement, i.e., the ratio between output and input noise to signal power ratio. The output noise to signal power ratio, denoted Γ, equals $\gamma\|\vec{e}\|^2/\|h\|^2$. The output noise to signal power corresponding to the MMSE solution of (7) is given by

$$\Gamma = \gamma h_\nu^T\Lambda_0^T(\underline{C}^T\underline{C})^{-1}\Lambda_0h_\nu / h_\nu^T\Lambda_0^T\Lambda_0h_\nu + o(\gamma)$$

The main contribution can be viewed as the Rayleigh ratio of $(\underline{C}^T\underline{C})^{-1}$ in the metric transformed by Λ_0.

Moreover, when some zeros are "close", the noiseless ideal setting has a large norm which would induces noise enhancement, still the smoothing factor balance the equalizer norm and thus the noise enhancement.

4. ACHIEVABLE PERFORMANCES

Given the lower bound of the input / output error power in terms of MMSE, we want to study the achievable performances of adaptive algorithms.

First, note that for the most popular equalization algorithm with training: Least Mean Square (LMS) algorithm (which is the stochastic gradient descent algorithm build from the MSE cost-function) the possible convergence points are the minima of the MSE cost-function $f(\vec{e})$. Its best acheivable performance are also these studied above. This algorithm is implemented in a Fractionally Spaced Equalizer setting, and called FSE-LMS.

In order to study blind adaptive algorithms, we set a general framework which is developped in the specific case FSE-CMA. FSE-CMA derives from the Constant Modulus Algorithm (see [11],[12]) implemented in a Fractionally Spaced case, see ([1]). The FSE-CMA updating equation is

$$\vec{e}(n+1) = \vec{e}(n) + \mu y(n)(r_2 - y^2(n))\vec{R}(n) \quad (8)$$

where $\vec{R}(n)$ is the NL-dimensional received signal and $y(n) = \vec{e}^T(n)\vec{R}(n)$ is the equalizer output. $r_2 = E[s^4]/E[s^2]$.

$g(\vec{e}) = E\left[(r_2 - y^2(n))^2\right]$ is the cost-function to be minimized. It can be proved ([13]) that the FSE-CMA cost-function under noisy conditions can be writen as the noise-free cost-function, denoted $g_0(\vec{e})$, added to a term equal to

$$g_\gamma(\vec{e}) = \gamma\|\vec{e}\|^2 \left(2(3\|h\|^2 - \rho) + 3\gamma\|e\|^2\right) \quad (9)$$

This assumes only condition 3 and independance between the source and noise signals. The additional term $g_\gamma(\vec{e})$ depends on the equalizer norm second and fourth order power. As for LMS, it can be viewed as a smoothing factor of the same power as the noise-free cost-function $g_0(\vec{e})$. Therefore, one should expect the noise to affect the steady-state performances in similar manner than for LMS. However, one should wonder how much additional MSE due to noise will appear in the case of FSE-CMA compared to that of FSE-LMS.

Simulations :
We compare the FSE-LMS and the FSE-CMA: (a) within condition 1, (b) without condition 1. We consider a 2-dimensional channel and a BPSK input signal ($s(n) = \pm 1$). The model for (a) is given above. For (b), the zeros of $c_1(z)$ are $(-1.4, -0.4)$, the zeros of $c_2(z)$ are $(1.1, -0.4)$. Simulations use a step-size equal to 10^{-4}, and the average of the MMSE at steady-state is estimated over 20000 iterations. In order to check only the lower MSE (we don't want to deal with eventual local minima here), the equalizers are initialized very close to the global minima settings. We consider a 4-taps equalizer.

The following curves display MMSE for FSE-LMS and FSE-CMA versus SNR. We see that FSE-LMS and FSE-CMA have very similar values in both cases. In case (a), Figure 4 shows that MMSE asymptotically converges to zero. However, in the case of common zero (b), Figure 5, MMSE converges to the lower non-avoidable error bound, which depends of the length of the equalizer.

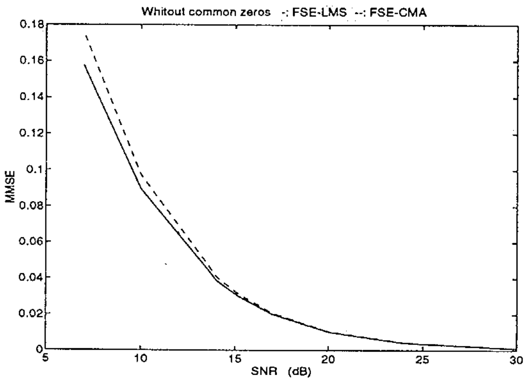

Figure 4: MMSE in case(a)

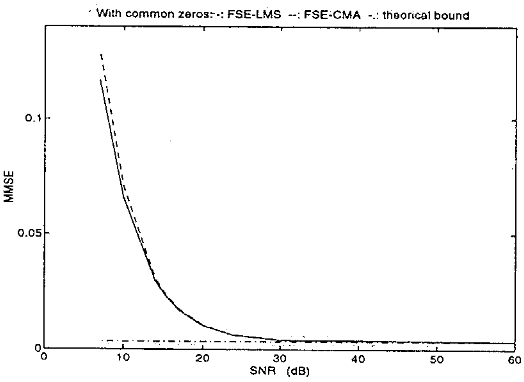

Figure 5: MMSE in case(b)

5. CONCLUSION

Equalizability : the best acheivable equalizer for a given channel, equalizer length and channel noise has been evaluated in terms of MMSE. In order to compare adaptive algorithms, we compare their MMSE to the equalizability power bound. A first study of FSE-CMA with respect to MMSE has been presented. It shows that the MMSE of FSE-CMA is very close to the optimal power bound. However, many questions remain. In particular, the effect of equalizer length on the MMSE of different algorithms is to be checked. Futhermore, our study concerns only the mean steady-state value and should be extended to MSE resulting from stochastic jitter around the mean solution.

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