Biorthogonal Wavelets and Rate-Distortion Algorithms

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Résumé

Les algorithmes qui déterminent le meilleur choix de quantificateurs pour un critère "débit-distorsion" dans un schéma de compression d'image par transformée sont souvent conçus pour des transformations orthogonales. Pour les utiliser avec des transformations biorthogonales, des précautions sont à prendre: cela nous conduit à étudier la non-conservation de l'énergie pour un banc de filtres biorthogonal et à étudier les performances "débit-distorsion" d'un schéma de compression de référence pour plusieurs transformations biorthogonales. Une méthode de normalisation des filtres est proposée, qui permet d'utiliser simultanément une transformation biorthogonale et un algorithme "débit-distorsion" de façon optimale ou presque.

Abstract

Algorithms which determine the best set of quantizers with reference to a "rate-distortion" criterion in a transform coding scheme are usually designed to deal with orthogonal transforms. When such algorithms are used with biorthogonal transforms, the non-orthogonality of these transforms might have to be taken into account: We study the non-orthogonality of biorthogonal transforms with reference to energy preservation and then the behaviour of different biorthogonal transforms in a realistic transform coding scheme for image compression. We propose a simple filter gain normalization technique which lets us use "rate-distortion" algorithms with biorthogonal transforms and yields results very close to optimum.

1 Introduction

While most of the transforms used in signal processing are orthogonal, some biorthogonal transforms have been introduced in the last few years, among which the most famous are the biorthogonal wavelet transforms. Orthogonal transforms have some nice properties, such as energy preservation, that are used e.g. in quantization procedures and bit allocation algorithms which optimize the system in a rate-distortion sense. These properties make the orthogonal transforms very attractive, but in the case of dyadic wavelets, orthogonality is non-compatible with phase-linearity, which is often required too. Thus it is desirable to be able to use simultaneously biorthogonal wavelets and rate-distortion algorithms.

In this paper we discuss this from an experimental point of view: this completes the theoretical approach presented in [2]. We observe the practical lack of an energy preservation property, discuss it with reference to the theoretical results, and propose a way to improve the quality of wavelet filter banks from this point of view without changing the other properties of the filter bank. This allows to use simultaneously biorthogonal transforms and rate-distortion algorithms yielding results very close to optimum.

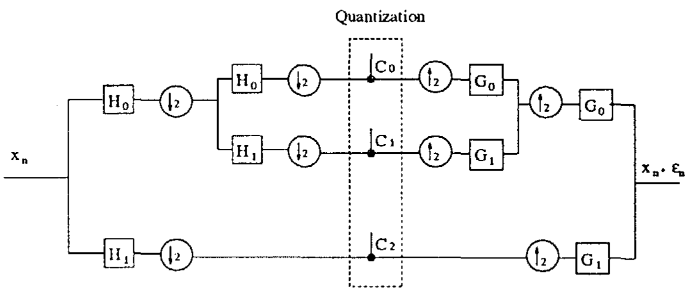

Figure 1: Example of a wavelet filter bank.

2 Rate-Distortion Algorithms for Transform Coding

We consider wavelet filter banks like the one in Figure 1. The analysis part consists in filtering and decimating the signal in each subband and in iterating this processing on the signal in the low-pass band. The quantization part is modelled by the addition of a quantization noise, whatever the algorithm used might be. The synthesis part consists in upsampling and filtering the signal in each subband and in adding all the subbands. We consider perfect reconstruction (PR) filter banks: Without any quantization error, the reconstructed signal is the original signal, up to a delay. Such PR filter banks may be orthogonal or not. In the latter

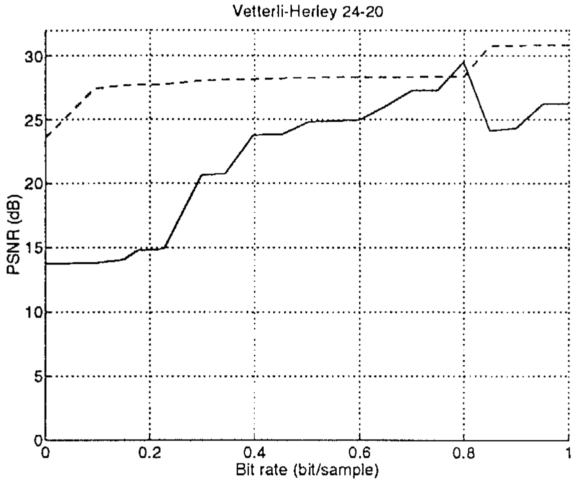

Figure 2: Predicted (dashed) and observed (solid) rate-distortion curves for biorthogonal Vetterli–Herley 24-20 filter bank [7], a filter bank which is far from orthogonality.

case, we call them biorthogonal filter banks. We use 1-D notations, but the N-D extension is straightforward.

We can sum up now the principle of rate-distortion algorithms [6]. The problem consists in choosing the quantizers for each subband. This choice is made in a rate-distortion sense. For all possible quantizers and for all subbands rates and distortions are computed. The algorithm attempts to choose the best combination in a rate-distortion sense [6]. For complexity reasons it is necessary to choose the best set without reconstructing the image. The resulting reconstruction error has to be computed through adding the subband distortions. This is true when the transform is orthogonal but untrue in the case of a biorthogonal transform. Figure 2 demonstrates the difference the PSNR computed as the sum of the subband distortions and the observed PSNR. The difference is easy to see. In addition the choice of the set of quantizers is clearly sub-optimal, because an optimal choice would not have lead to a decreasing curve (between rates 0.8 and 0.85)! This rate-distortion curve shows that a further study is necessary before using rate-distortion algorithm with a biorthogonal filter bank.

3 Biorthogonal Filter Banks and Energy Preservation

Let us summarize briefly the theoretical results on biorthogonal filter banks and energy preservation as obtained in [2]. These results hold for parallel filter bank formulation. The application to wavelet iterated filter banks is done by considering an equivalent filter bank.

There are two approaches to this problem. The first one consists in computing bounds which control the energy preservation of any signal. The second one consists in computing the energy preservation gain for a class of signals defined by its power density function.

3.1 Computing general bounds

For rate-distortion algorithms we assume rate and distortion to be additive over the subbands [5, 6]. In other words, we want the sum of the subband square errors to be close to the reconstruction square error. That means that there exist two constants A and B, close to 1, so that, whatever the quantization might be,

$$A \sum_{j=0}^{M-1} \sum_{n} c_{j,n}^2 \leq \sum_{n} \varepsilon_n^2 \leq B \sum_{j=0}^{M-1} \sum_{n} c_{j,n}^2, \qquad (1)$$

where ε denotes the reconstruction error (in the time domain), c_j the quantization error in the subband j, and M the total number of subbands.

We are able to calculate A and B by considering the reconstruction error in the frequency domain (cf. Fig. 1),

$$E(\omega) = \sum_{j=0}^{M-1} C_j(M\omega) G_j(\omega). \qquad (2)$$

The corresponding energy is given by,

$$\int_0^{\pi} |E(\omega)|^2 \, d\omega = \int_0^{\pi} \langle C(\omega), S(\omega) C(\omega) \rangle \, d\omega \qquad (3)$$

where we introduced the operator

$$S(\omega) = (S_{ij}(\omega))_{0 \leq i \leq M-1,\, 0 \leq j \leq M-1} \qquad (4)$$

$$S_{ij}(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} G_i\left(\frac{\omega + 2k\pi}{M}\right) G_j^{\star}\left(\frac{\omega + 2k\pi}{M}\right) \qquad (5)$$

Then the bounds A and B holding for any signal are given by the infimum and the supremum of the spectrum of S [2].

3.2 The white noise case

In the particular case of a white noise model, the expectation of the error turns out to be very simple,

$$\sum_{k=0}^{M-1} N_k \|G_k\|^2. \qquad (6)$$

This is just a weighted version of the distortion to be taken into account in the orthogonal case. However it is possible to make all weights $\|G_k\|^2$ equal, which still allows to use the rate-distortion algorithm as a "black box".

The basic idea we propose follows. When designing a two-band biorthogonal filter bank, one can always multiply the analysis low-pass-filter and the synthesis high-pass filter by a given scalar λ and divide the other filters by λ. This does not change any of the properties of the filter bank i.e. perfect reconstruction, smoothness of the iterated filter (regularity), flatness, frequency selectivity, coding gain,etc. However this scaling changes the non-orthogonality. When considering an iterated filter bank, we will optimize the normalization scalar for each cell. We want all the filters of the equivalent parallel filter bank to have the same energy.

Let us provide the formulation for the proposed scheme. We consider a biorthogonal two-band filter bank with a synthesis low-pass filter G_0 and a synthesis high-pass filter G_1 and a J-times iterated filter bank. Let $u_1, \ldots, u_{J+1}$ denote the energies of the synthesis filters of the equivalent parallel filter bank. Let the iterations be done with normalization scalars $\lambda_1 \ldots \lambda_J$. We obtain:

$$\begin{aligned} u_1(\lambda) &= \lambda_1 u_1, \\ u_2(\lambda) &= \frac{\lambda_2}{\lambda_1} u_2, \\ u_3(\lambda) &= \frac{\lambda_3}{\lambda_2 \lambda_1} u_3, \\ &\vdots \\ u_J(\lambda) &= \frac{\lambda_J}{\lambda_{J-1} \ldots \lambda_1} u_J, \\ u_{J+1}(\lambda) &= \frac{1}{\lambda_J \ldots \lambda_1} u_{J+1}. \end{aligned} \tag{7}$$

The choice of the normalization scalars is then straightforward,

$$\lambda_J = \sqrt{\frac{u_{J+1}}{u_J}} \tag{8}$$

$$\lambda_k = \sqrt{\lambda_{k+1} \frac{u_{k+1}}{u_k}}, \quad k = J-1 \ldots 1 \tag{9}$$

In practice the λ values are always similar to each other along the subbands. Then the equivalent parallel synthesis filters have the same energy: as the reconstruction error is theoretically the subband distortion with a gain, it is easy to be predicted.

4 Further analysis of rate-distortion curves

The results of the previous section are useful in analyzing the use of rate-distortion algorithms with biorthogonal transforms. In the following experimental analysis we will use the algorithms presented in [3, 4], with Lena 256, scalar quantization, Huffmann lossless coding, and 30 quantizers available.

The normalization procedure is efficient. This is depicted in Figure 3 showing that the resulting PSNRs are much better than the results in Figure 2 where the proposed scheme is not used (note that both curves

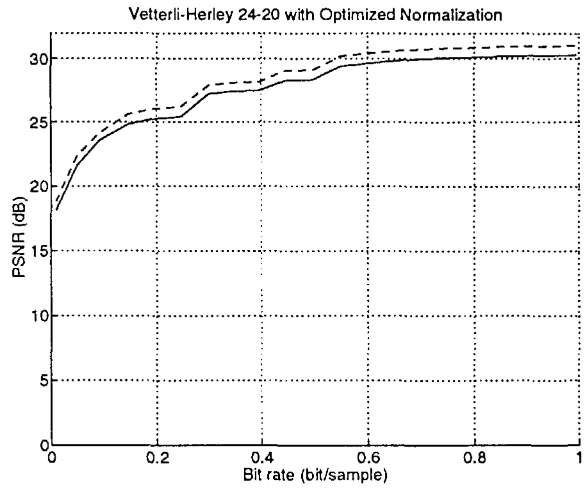

Figure 3: Predicted (dashed) and observed (solid) rate-distortion curves for the Vetterli-Herley 24-20 filter bank with an optimal choice of normalization scalars.

are parallel). The observed curve is then an increasing function, and the prediction looks more accurate. Assuming a white noise model, the prediction would be perfect up to a simple gain. As the bit rate increases, the prediction comes closer to this model. Such parallel predicted and observed rate-distortion curves also arise with other examples (after normalization) such as Onno's biorthogonal wavelets [3]. With "near-orthogonal" [2] wavelets, the optimal normalization is not even needed, as we saw with Burt-Cohen-Daubechies 5-7 filter bank [1]. Both low-pass filters have very similar frequency responses: Intuitively we may say that the filter bank is nearly-orthogonal. With 4 iterations the bound B is 1.23. In the worst case (which is not realistic at all) the resulted difference in PSNRs might be 2 dB. In practice the difference between predicted and observed PSNRs is negligible (0.03 dB).

The parallel curves in themselves self do not prove that the use of the rate-distortion algorithm in the subband domain is optimal with the corresponding biorthogonal transforms. It only proves the quality of the prediction for some sets of quantizers. Other sets of quantizers may provide worse results than the chosen ones in the subband domain while providing better results in the reconstruction error sense. Such behaviour would not be reflected in the curves of Figure 3. If we want to know whether the chosen set of quantizer is the optimal one, we have to compare it with all the possible ones. Owing to the large number of combinations involved, such an exhaustive comparison is not practical. Therefore we chose randomly selected sets of quantizers, and we compared the resulted reconstruction errors. This experiment is depicted in Figure 4. Each cross corresponds to a random set of quantizers. In few cases they actually provide better results in terms

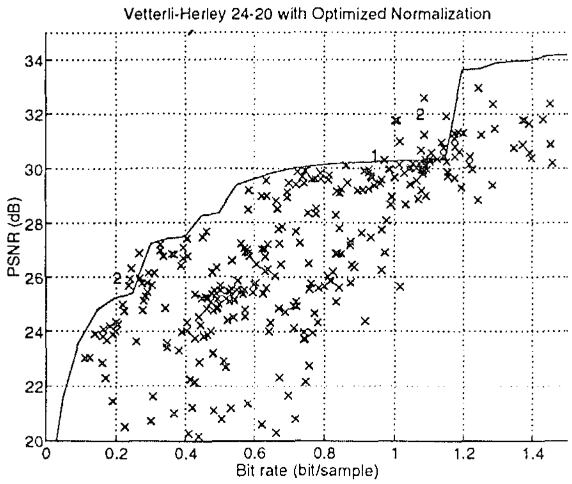

Figure 4: Observed rate-distortion curve for the Vetterli–Herley 24-20 filter bank after optimized normalization. The crosses correspond to other choices of quantizers. The crosses above the curve are due to the sub-optimality of the rate-distortion algorithm but not to the non-orthogonality of the transform.

of reconstruction error, which means sub-optimality of the set selection algorithm. When one has a closer look to the predicted and observed PSNRs for these points above the curve, two kinds of such situations are to be observed:

1. The sub-optimality is due to the non-orthogonality of transforms, because the predicted PSNR of the chosen set is higher than the one of the random set, while the observed PSNR of the random set is higher. However for such cases the difference is very small, less than 0.1 dB. An example of this situation has label 1 in Figure 4.

2. The sub-optimality is due to the sub-optimality of the rate-distortion algorithms. Even in the orthogonal case the practical rate-distortion algorithms may be sub-optimal, because it cannot test all possible quantizers combinations. For such cases the sub-optimality of the chosen set has nothing to do with the use of a biorthogonal transform. When the difference between the chosen and the random sets of quantizers is high (e.g. 1 dB or more), this is due to this phenomenon. Examples of this situation have label 2 in Figure 4.

We may conclude that practically the result of the rate-distortion algorithm in the biorthogonal case is as good as in the orthogonal one: When the optimal combination of quantizers is selected in the sense of maximizing the predicted PSNR for a given bit rate, the combination is also optimal (or very close to optimum, e.g. 0.05 dB) in the sense minimizing the reconstruction error for the given bit rate.

5 Conclusion

We have discussed the behaviour of the rate-distortion algorithm for biorthogonal transforms. Depending on the filters used, the rate-distortion algorithm can be far from optimum. However, through the normalization scalars optimization, we provided a simple way to make it very close to optimum.

A discussion on the orthogonality criterion may follow: Should we look for a good transform among the orthogonal transforms?

- For stationnary signals the optimal transform is the Karhunen-Loeve transform, which is orthogonal. However images are non-stationnary.

- For the use of rate-distortionalgorithms energy preservation may be needed. Our study shows that under certain conditions, rate-distortion algorithms may be used optimally (or close) with biorthogonal transforms.

The discussion is still open but this work on rate-distortion optimality extends the application field for biorthogonal transforms.

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