A NEW POLE ALLOCATION PROCEDURE FOR HIGH ORDER RECURSIVE FILTERS

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RÉSUMÉ

Il faut souvent utiliser une grande taille de données ou de coefficients pour les filtres d'ordre élevé, réalisés avec une structure conventionnel. Il a été démontré que nouvelles structures hybrides sont largement meilleures à cet égard.

Cet article présente une nouvelle procédure d'allocation des pôles pour la synthèse des filtres d'ordre élevé. Nous allons montrer que cette procèdure est une étape clé du processus de synthèse de ces filtres pour avoir des coefficients au numérateur de taille minimale. L'exemple donné révèle l'ampleur des amélioration possibles.

ABSTRACT

Realisations of high order filters using conventional filter structures often require excessive wordlengths, either for the coefficients or for the data. New hybrid structures have been shown to be considerabley better in these respects.

This paper introduces a new pole allocation procedure for synthesising high order recursive filters with hybrid structures. This is shown to be a key stage of the synthesis process for obtaining hybrid structure high order filter realisations with minimum numerator coefficient wordlengths. An example is given which indicates the extent of the improvement over conventional structures.

1 INTRODUCTION

It is usual to implement a recursive digital filter as an interconnection of first and second order sections. This is primarily done to minimise the sensitivity of the filter to coefficient quantisation. Normally the sections are connected either in cascade or in parallel. The cascade and parallel filter structures work well for low order filters (eg less than 14th). However for higher orders, the use of extremely large data or coefficient wordlengths becomes necessary.

New hybrid filter structures [1, 2], have been devised to overcome these practical difficulties with implementing high order recursive filters. The advantage of the new structures is that they can be used to minimise the excessive data and coefficient wordlengths required for high order recursive filter realisations. The trade-off for this is an increase in the computational complexity in the synthesis of such filters.

The concept behind the hybrid filter structures is to partition a high order recursive filter into lower order subfilters which are easier to realise. The overall filter is formed by connecting the subfilters together in cascade or in parallel. Since each subfilter consists of more than one section, there are many ways in which the poles of the filter transfer function can be allocated to the subfilters. We have found that the required data and coefficient wordlengths for a filter implementation are largely dependant on the pole allocation arrangement used.

There are two classes of the hybrid filter structures: 'Cascade Interconnected Subfilters'; and 'Parallel Interconnected Subfilters'. This paper examines how high order 'Parallel Interconnected Subfilters' (PIS) class filters with minimum coefficient wordlengths can be synthesised.

2 PIS STRUCTURES

The general form of all PIS class hybrid structures is illustrated in figure 1. Each structure in this class consists of low order subfilters that are connected in parallel to give the required filter transfer function. The subfilters can be realised with any classical structure - parallel form, cascade form, and lattice are good examples. So the synthesis process is effectively broken down into two stages:
(i) Partition the filter transfer function into subfilters,
(ii) Synthesise the subfilters using conventional methods.

Since stage (ii) is based on well established methods, we are only concerned with stage (i) here.

The filter is partitioned into subfilters by obtaining an incomplete partial fraction decomposition of the filter transfer function, thus:

$$H(z) = \frac{u}{q}(z) = \sum_{k=0}^{F-1} \frac{p'_k}{q'_k}(z) \qquad (2.1)$$

where F = number of subfilters and $q'_k(z)$ are known real factors of $q(z)$, with degrees $n_k > 2$.

Efficient methods for obtaining this decomposition, based on an algorithm due to Shamash [3], were identified by Sandler [1]. Enhancements to these algorithms are still under development [4, 5].

3 COEFFICIENT WORDLENGTHS

Parallel form realisations of high order filters frequently require excessive numerator coefficient wordlengths, because they often have a large range of numerator coefficient magnitudes. It also necessitates the use of longer data wordlengths in order to maintain the original dynamic range of the data. This problem can also occur with PIS structures if the poles are not allocated appropriately.

The 'frequency spacing' procedure for allocating poles to subfilters was proposed in [1], which maximises frequency spread of the poles in each subfilter. The aim of this was to maximise the signal to quantisation noise ratio in PIS realisations of high order recursive filters, at a given data wordlength. Results show that this procedure often yields lower ranges of numerator coefficient magnitudes in PIS realisations than in parallel form realisations. However, it does not usually give PIS realisations with **minimum** coefficient magnitude ranges.

As a way of measuring the range of coefficient magnitudes, we define the range factor Φ as:

$$\Phi = \frac{largestmagnitude}{smallestmagnitude} \tag{3.1}$$

For any given filter, the pole allocation arrangement that yields the realisation with the minimum range factor cannot yet be predicted. Hence, in order to obtain the optimum realisation in this sense, each possible partition must be synthesised. The optimum realisation can then be selected from the set.

This process requires a very large number of synthesis iterations. The exact number for any given order depends on the nature of the structure, but for example, a PIS structure with subfilters of equal order requires

$$Iterations = \frac{[\frac{ORDER}{2}]!}{F(S!)^F} \tag{3.2}$$

iterations, assuming the filter $ORDER$ is even, and where: F = number of subfilters; and S = order of subfilters.

Given that we are interested in filter orders in excess of 100, this number of iterations is phenonomal. However, by appealing to the relevent theory, some of the partitions can be predicted as being non-optimum, and can therefore be excluded from the process.

4 THEORY

The relationship between PIS class structures and the parallel form gives insight into how some of the non-optimum pole arrangements can be predicted.

Assume that the recursive (pole/zero) filter transfer function is of the form

$$H(z) = \frac{u}{q}(z) \tag{4.1}$$

where

$$q(z) = \prod_{\gamma=1}^{n} q_\gamma(z) \tag{4.2}$$

$q_\gamma(z)$ are the lowest order factors of $q(z)$ with only real coefficients - ie real poles and complex conjugate pairs of poles,

and $u(z)$ is a polynomial in z which represents the zeros of the filter.

The parallel equivalent transfer function is given as

$$H(z) = p_0(z) + \sum_{\gamma=1}^{n} \frac{p_\gamma}{q_\gamma}(z) \tag{4.3}$$

where the order of each $p_\gamma(z)$ is one less than that of the corresponding $q_\gamma(z)$ (for $\gamma > 0$),

and $p_0(z)$ only exists if the order of $u(z)$ is not less than the order of $q(z)$.

The subfilter transfer functions of an equivalent PIS class filter can be formed by cross-multiplying groups of terms in the summation of equation (4.3) - although we would not do this in practice because it generates large errors. Each subfilter denominator $q'_k(z)$ consists of groups of $q_\gamma(z)$ such that

$$q'_k(z) = \prod_{\gamma=i}^{L_k+i-1} q_\gamma(z) \tag{4.4}$$

where L_k =number of complex pairs and real poles in the subfilters for $k = 0, 1, ..., F-1$, and $i = \sum_{\alpha=0}^{k} L_k$.

Each subfilter numerator $p'_k(z)$ consists of a linear combination of $p_\gamma(z)$ and $q_\gamma(z)$, thus

$$p'_k(z) = \sum_{\gamma=i}^{L_k+i-1} \frac{p_\gamma {q'}_k}{q_\gamma}(z) \tag{4.5}$$

With this relationship, some of the non-optimum pole arrangements for any given filter can be identified from the parallel form transfer function (4.3). These arrangements are summarised as follows.

- When all, or nearly all of the $q_\gamma(z)$ whose $p_\gamma(z)$ have the largest coefficient magnitudes are in the same subfilter.
- When all, or nearly all of the $q_\gamma(z)$ whose $p_\gamma(z)$ have the smallest coefficient magnitudes are in the same subfilter.

Hence, if we obtain the parallel form equivalent transfer function prior to synthesis, the optimisation process can be reduced accordingly.

For example, if a given PIS structure has say, two 12th order subfilters - ie six sections each - then no subfilter should have more than three of the $q_\gamma(z)$, whose corresponding $p_\gamma(z)$ have the largest coefficient magnitudes allocated to the same subfilter. Similarly for the $q_\gamma(z)$ whose corresponding $p_\gamma(z)$ have the smallest coefficient magnitudes. A combinatorial analysis [6] of this scenario rules out a total of 50 out of the 461 non-optimum partition arrangements.

The combinatorial analysis becomes much more involved as we increase the complexity and order of filter structures. However, for filter orders upto 50 we would normally expect to rule out about 10 per cent of the total number of possible partitions.

5 PROCEDURE

The 'frequency spacing' allocation procedure does not usually yield realisations with the minimum range factor Φ. However, the range factors are often sufficiently small. In these cases, the optimisation process can be bypassed altogether. Thus, a suitable allocation procedure is as follows.

(i) The filter transfer function is specified with $u(z)$ fully expanded and $q(z)$ as the list of $q_\gamma(z)$.

(ii) Decide on the number of subfilters that the resultant PIS structure filter will have - based on the filter order.

(iii) Synthesise the PIS structure filter using the 'frequency spacing' allocation procedure. If the range factor Φ is too large then continue. Otherwise finish.

(iv) Compute the parallel equivalent transfer function (equation 4.3).
(v) Determine which arrangements are guaranteed to be no good from the parallel form transfer function.
(vi) Synthesise the rest of the partitions, and select the best.

6 EXAMPLE

The above procedure was tested with a variety of 'synthetic' and 'real-life' filters with orders ranging from 16 to 48. We have used one of these filters for the following example. The filter was generated using musical instrument analysis/synthesis software [7]. It is a reduced order version of one of a sequence of filters which collectively form a model of a tom-tom drum.

The filter is all-pole - ie $u(z) = 1$ - and the $q_\gamma(z)$ are given as:
$q_1(z) = (1.0 - 1.965699z^{-1} + 0.966944z^{-2})$;
$q_2(z) = (1.0 - 1.962799z^{-1} + 0.966944z^{-2})$;
$q_3(z) = (1.0 - 1.942954z^{-1} + 0.966944z^{-2})$;
$q_4(z) = (1.0 - 1.831617z^{-1} + 0.969677z^{-2})$;
$q_5(z) = (1.0 - 1.515992z^{-1} + 0.966944z^{-2})$;
$q_6(z) = (1.0 - 0.941678z^{-1} + 0.975156z^{-2})$;
$q_7(z) = (1.0 + 0.014895z^{-1} + 0.986160z^{-2})$;
$q_8(z) = (1.0 + 1.981327z^{-1} + 0.983403z^{-2})$.

A z plane diagram of the filter is illustrated in figure 2.

The PIS structure filter was chosen to have 2 subfilters, each with an order 8. Hence for this example, the total number of partitions is:

$$\frac{8!}{2(4!)^2} = 35 \qquad (6.1)$$

Since this is a relatively small amount of partitions, the full unreduced optimisation process was performed in the first instance, so that all of the realisations could be observed. The reduced optimisation process excluded 8 partitions.

The numerators of the parallel equivalent $p_\gamma(z)$, were computed as:
$p_1(z) = (212916.83 - 183790.67z^{-1})$;
$p_2(z) = (-244362.06 + 212536.99z^{-1})$;
$p_3(z) = (31518.370 - 29029.214z^{-1})$;
$p_4(z) = (-60.519375 + 266.93855z^{-1})$;
$p_5(z) = (-11.144283 + 16.038683z^{-1})$;
$p_6(z) = (-0.4768660 + 0.6411380z^{-1})$;
$p_7(z) = (0.00282510 + 0.02304710z^{-1})$;
$p_8(z) = (0.00160873 + 0.00140246z^{-1})$.

The worst pole allocation arrangement was:
$q_1'(z) = q_1 \cdot q_2 \cdot q_4 \cdot q_5(z)$; $q_2'(z) = q_3 \cdot q_6 \cdot q_7 \cdot q_8(z)$,

with numerators:
$p_1'(z) = (-31516.668 + 197117.25z^{-1} - 543344.68z^{-2}$
$+856993.54z^{-3} - 835755.28z^{-4} + 503782.40z^{-5}$
$-173589.43z^{-6} + 26317.920z^{-7})$;
$p_2'(z) = (31517.668 + 4208.9196z^{-1} + 3882.6633z^{-2}$
$+32295.881z^{-3} - 25425.227z^{-4} + 790.42030z^{-5}$
$+591.07870z^{-6} - 27452.0794z^{-7})$.

The best pole allocation arrangement was:
$q_1'(z) = q_1 \cdot q_2 \cdot q_4 \cdot q_8(z)$; $q_2'(z) = q_3 \cdot q_5 \cdot q_6 \cdot q_7(z)$,

with numerators:
$p_1'(z) = (-31505.751 + 86852.821z^{-1} - 21553.665z^{-2}$
$-142375.56z^{-3} + 132745.97z^{-4} + 29737.813z^{-5}$
$-80608.560z^{-6} + 26751.346z^{-7})$;
$p_2'(z) = (31506.751 - 105971.17z^{-1} + 206943.25z^{-2}$
$-275340.04z^{-3} + 271433.26z^{-4} - 196503.21z^{-5}$
$+97230.683z^{-6} - 26977.911z^{-7})$.

The 'frequency spacing' allocation procedure arrangement was:
$q_1'(z) = q_1 \cdot q_3 \cdot q_5 \cdot q_7(z)$; $q_2'(z) = q_2 \cdot q_4 \cdot q_6 \cdot q_8(z)$,

with numerators:
$p_1'(z) = (244424.05 - 1055324.5z^{-1} + 2155715.1z^{-2}$
$-2873085.3z^{-3} + 2816970.1z^{-4} - 2016774.8z^{-5}$
$+924922.43z^{-6} - 196213.70z^{-7})$;
$p_2'(z) = (-244423.05 + 406388.15z^{-1} + 37186.929z^{-2}$
$-629941.02z^{-3} + 586477.22z^{-4} + 21932.780z^{-5}$
$-393505.81z^{-6} + 197884.88z^{-7})$.

The numerator coefficients are quite diverse from one arrangement to the next. The best arrangement had a range factor Φ, of 12.77 - a significant improvement on the range factor of about 174240080 for the parallel form. The 'frequency spacing' partition falls somewhere between the optimum and the worst in this example, having a range factor of 130.99.

Even the worst arrangement for this example was quite an improvement on the parallel form, having a range factor of 1449.88. However, this is quite a low order filter. As we progress to higher order filters, the diversity becomes more extreme and hence more of a problem.

7 OTHER CONSIDERATIONS

The next stage of the research will investigate the effects of pole allocation on filter performance. The performance of the implementation of the best of the three arrangements given in the example, compared very favourably, having a lower frequency response error than the other two arrangements. Also, as expected, its SQNR was not quite as high as the SQNR of the 'frequency spacing' arrangement, but was higher than the SQNR of the worst partition arrangement. Since the example filter is the only one that has been tested at present, the results are inconclusive. It is anticipated that more conclusive results on performances will be obtained in the near future.

8 CONCLUSIONS

Large ranges of numerator coefficient magnitudes often occur in parallel form realisations of high order filters, forcing the use of excessive coefficent wordlengths in their implementation. The PIS filter structures are a class of hybrid filter structures which were devised to overcome both this problem and also the requirement for excessive **data** wordlengths experienced in cascade form realisations. However, care must be taken over the allocation of poles within the structures for them to be effective in these respects.

The pole arrangement yielding the minimum numerator coefficient magnitude range cannot yet be predicted for any given filter. However, certain non-optimum pole arrangements can be predicted for a filter, by referring to the parallel form equivalent realisation. Hence the iterative synthesis process, which synthesises the realisation with the minimum coefficient magnitude range for any given filter, can be reduced accordingly.

Future work involves the comparative examination of filter performance, in terms of frequency response error and SQNR, for a variety of examples of high order recursive filters.

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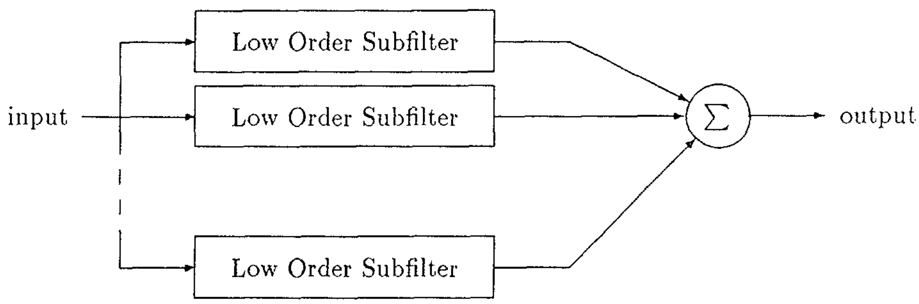

Figure 1: General Form of the PIS Category Structure

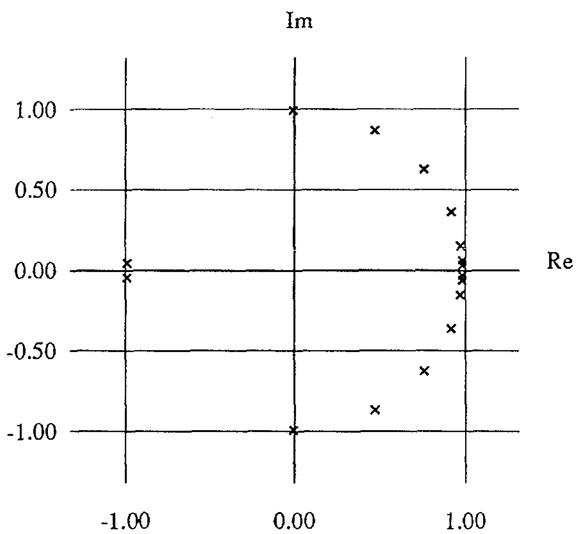

Figure 2: Z Plane Diagram of Example Filter.