NON-PARAMETRIC DETECTION OF TYPICAL GRAVITATIONAL WAVES SIGNALS

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RÉSUMÉ

Dans cet article nous abordons le problème de la détection des ondes gravitationelles noyés dans un bruit normal blanc avec densité spectrale inconnue. En utilisant la strategie de le Rapport de Vraisemblance Generalisé, on presente un nouvel détecteur non-paramétrique, et on montre qu'il peut être réalisé à l'aide d'un algorithm de Fast Fourier Transform. Enfin, on montre que cet détecteur obtient une probabilité de fausse alarme constante par rapport à la densité spectrale du bruit.

ABSTRACT

In this paper we handle the detection of Gravitational Waves in the presence of white Gaussian noise with unknown Power Spectral Density. Based upon a Generalized Likelihood Ratio optimization strategy, we present a new non-parametric detector, showing that it lends itself to implementation through Fast Fourier Transform algorithms. Finally, we show that it achieves Constant False Alarm Rate with respect to the noise power spectral density.

1. INTRODUCTION

In its simplest form, namely neglecting tidal effects, post-Newtonian corrections, doppler shifts (due to the earth motion), and taking the eccentricity of the orbit to be zero, the noise-free response of the detector to a Gravitational Wave (GW) produced by coalescing binaries is

$$s(t) = F_+ h_+(t) + F_\times h_\times(t) \tag{1}$$

where the subscripts "+" and "×" denote the polarization of the two components [1] and where

$$h_+(t) = B(t)(1+\cos^2 i)\cos\left[2\pi\int_{t_0}^{t} f(\xi)d\xi + \phi\right] \tag{2}$$

$$h_\times(t) = 2B(t)\cos i \sin\left[2\pi\int_{t_0}^{t} f(\xi)d\xi + \phi\right] \tag{3}$$

with i the angle between the line-of-sight of the detector and the orbital plane of the binary system, t_0 the arrival time, and ϕ a phase term. As to $B(t)$ and $f(t)$, they are given by

$$B(t) = \frac{2\mu G}{rc^4}\left[GM\pi f(t)\right]^{2/3} \tag{4}$$

$$f(t) = f_0\left(1 - \frac{t-t_0}{\tau}\right)^{-3/8} \tag{5}$$

wherein μ and M are the reduced and the total mass of the binary system, respectively, r is its distance from the Earth, while τ is the signal duration, which in turn is related to μ and M. In conclusion, the useful signal can be cast in the compact form

$$s(t) = HA(t,t_0,\tau)\cos\left[\beta(t,t_0,\tau) + \Phi\right] \tag{6}$$

The observable waveform, $r(t)$ say, is a corrupted version of (6), due to the presence of an additive noise component. Thus, the detector is to implement the *statistical* hypothesis test:

$$\begin{array}{lll} H_1 : & r(t) = s(t; H, \Phi, t_0, \tau) + n(t) & 0 \le t \le T \\ H_0 : & r(t) = n(t) & 0 \le t \le T \end{array} \tag{7}$$

where $n(t)$ is a sample function from the noise process, which is assumed to be Gaussian, and T is the observation interval.

If the signal to be detected were completely known, the optimum - in the Neyman-Pearson sense - receiver would consist of a plain matched filter, whose sampled output should be compared to a suitable detection threshold. However, the signal contains four unknown parameters, such as the amplitude H, the phase Φ, the arrival time t_0 and the duration τ. This implies that the hypothesis H_1 in (7) is *composite*: A suitable strategy for circumventing the a-priori uncertainty as to the signal parameters is the Generalized Likelihood Ratio Test (GLRT), which has been already presented in [2] under the assumption that the noise spectral characteristics are perfectly known.

In real situations there is not perfect knowledge as to the noise due to the inherent difficulties in performing signal-free measurements. We consider here the case that the noise is *white*, but with unknown power spectral density (PSD) $\mathcal{N}_0/2$, whence both hypoteses in (7) are composite.

The final aim is to obtain a non-parametric detection structure, namely wherein a-priori knowledge of the noise PSD is not required either for evaluating the test statistic or for setting the threshold level. The paper is organized as follows: in next section, we borrow some known arguments from the statistical detection theory and use them to obtain the scheme of an optimized parameter-free detector, while in section 3 we present a completely numerical implementation of such a structure, showing that all of the relevant statistics may be efficiently calculated via a digital computer. Finally, section 4 contains concluding remarks and hints for future research.

2. DETECTOR DESIGN

Since the noise is Gaussian and white, its projections along the versors of any orthonormal basis of the space $L^2(0,T)$ of signals with finite energy in $(0,T)$ are independent and identically distributed Gaussian random variates. Thus, denoting by $r_k, k = 0, 1, \ldots$ the projections of the received signal along the versors of an arbitrary orthonormal basis, the continuous-time decision problem (7) can be reduced to a discrete one in the form

$$\begin{array}{lll} H_1 & : & r_k = s_k + n_k \\ H_0 & : & r_k = n_k \end{array} \tag{8}$$

wherein $s_k = s_k(H, \Phi, t_0, \tau)$ and n_k denote the projections of the signal and of the noise, respectively. If $\boldsymbol{r}$ is the vector of the first N projections of the received signal, then the likelihood functional can be written as

$$\Lambda\left[r(t); \mathcal{N}_0, H, \Phi, t_0, \tau\right] = \lim_{N\to\infty} \frac{f_{\boldsymbol{r};\mathcal{N}_0,H,\Phi,t_0,\tau|H_1}(\boldsymbol{r})}{f_{\boldsymbol{r};\mathcal{N}_0|H_0}(\boldsymbol{r})} \tag{9}$$

As expected, the functional depends upon the unknown noise and signal parameters. In order to circumvent the uncertainty as to $\mathcal{N}_0$, H and Φ, we resort to the GLRT, namely to the test

$$\frac{\max_{H,\Phi,t_0,\tau,\mathcal{N}_0} f_{\boldsymbol{r};\mathcal{N}_0,H,\Phi,t_0,\tau|H_1}(\boldsymbol{r})}{\max_{\mathcal{N}_0} f_{\boldsymbol{r};\mathcal{N}_0|H_0}(\boldsymbol{r})} \underset{H_0}{\overset{H_1}{\gtrless}} \Lambda_0 \tag{10}$$

Thus the test statistic is obtained by replacing in the likelihood functional $\Lambda\left[r(t); \mathcal{N}_0, H, \Phi, t_0, \tau\right]$ the unknown parameters by their respective maximum likelihood (ML) estimates under both hypotheses.

At first, we leave aside the problem of maximizing with respect to τ and t_0. Under H_0 hypothesis, and assuming that the expansion basis is real, $\boldsymbol{r}$ is a real zero-mean Gaussian random vector whose entries are independent and identically distributed with variance $\mathcal{N}_0/2$. Accordingly, the logarithm of its probability density function (pdf) is maximum at

$$\widehat{\mathcal{N}_0/2} = \frac{1}{N}\|\boldsymbol{r}\|^2 \tag{11}$$

Under H_1, instead, we have to solve the following system of simultaneous equations:

$$\begin{cases} \frac{\partial}{\partial X}\left[-\frac{N}{2}\ln \pi\mathcal{N}_0 - \frac{1}{\mathcal{N}_0}\|\boldsymbol{r} - X\boldsymbol{s}_c + Y\boldsymbol{s}_s\|^2\right] = 0 \\ \frac{\partial}{\partial Y}\left[-\frac{N}{2}\ln \pi\mathcal{N}_0 - \frac{1}{\mathcal{N}_0}\|\boldsymbol{r} - X\boldsymbol{s}_c + Y\boldsymbol{s}_s\|^2\right] = 0 \\ \frac{\partial}{\partial \mathcal{N}_0}\left[-\frac{N}{2}\ln \pi\mathcal{N}_0 - \frac{1}{\mathcal{N}_0}\|\boldsymbol{r} - X\boldsymbol{s}_c + Y\boldsymbol{s}_s\|^2\right] = 0 \end{cases} \tag{12}$$

wherein $\|\cdot\|$ denotes Euclidean norm, $X = H\cos\Phi$ and $Y = H\sin\Phi$, while $\boldsymbol{s}_c$ and $\boldsymbol{s}_s$ are the vectors of the first N projections of $s_c(t) = A(t)\cos\beta(t)$ and $s_s(t) = A(t)\sin\beta(t)$ along the basis, respectively: thus, the vector $\boldsymbol{s} = X\boldsymbol{s}_c - Y\boldsymbol{s}_s$ represents the first N projections of the signal $s(t)$ along the basis.

The solution is

$$\widehat{\mathcal{N}_0/2} = \frac{1}{N}\|\boldsymbol{r}_N - \widehat{\boldsymbol{s}_N}\|^2 \tag{13}$$

where $\widehat{\boldsymbol{s}_N} = \widehat{X}\boldsymbol{s}_c - \widehat{Y}\boldsymbol{s}_s$, with $\widehat{X}$ and $\widehat{Y}$ the ML estimates of X and Y, respectively. Moreover, denoting by Q_c and Q_s the energies of $\boldsymbol{s}_c$ and $\boldsymbol{s}_s$, respectively, and introducing the dot product $Q_{cs} = Q_{sc} = <\boldsymbol{s}_c, \boldsymbol{s}_s>$ the ML estimates of X and Y are written as

$$\begin{cases} \widehat{X} = \dfrac{\frac{<\boldsymbol{r},\boldsymbol{s}_c>}{Q_c} - \frac{Q_{cs}}{Q_cQ_s}<\boldsymbol{r},\boldsymbol{s}_s>}{\left(1 - \frac{Q_{cs}^2}{Q_cQ_s}\right)} \\ \widehat{Y} = -\dfrac{\frac{<\boldsymbol{r},\boldsymbol{s}_s>}{Q_s} - \frac{Q_{cs}}{Q_cQ_s}<\boldsymbol{r},\boldsymbol{s}_c>}{\left(1 - \frac{Q_{cs}^2}{Q_cQ_s}\right)} \end{cases} \tag{14}$$

For sufficiently high N, it can be shown that $Q_c = Q_s$ and $Q_{cs} = 0$, then the previous equations simplify to

$$\begin{cases} \widehat{X} = \frac{1}{Q_c} < \boldsymbol{r}, \boldsymbol{s}_c > \\ \widehat{Y} = -\frac{1}{Q_c} < \boldsymbol{r}, \boldsymbol{s}_s > \end{cases} \tag{15}$$

Direct substitution of the above results into the likelihood functional yields, for the GLR:

$$\Lambda\left[r(t); \widehat{\mathcal{N}_0}, \widehat{H}, \widehat{\Phi}, \widehat{t_0}, \widehat{\tau}\right] = \max_{t_0,\tau} \lim_{N\to\infty} \left(\frac{\frac{\|\boldsymbol{r}\|^2}{N}}{\frac{\|\boldsymbol{r}-\widehat{\boldsymbol{s}}\|^2}{N}}\right)^{N/2} \tag{16}$$

Notice that, in principle, the functional (16) is not ensured to converge, since it is not a likelihood ratio, whence the Grenander's convergence theorem [3] cannot be directly applied. However, it suggests a sub-optimum detection structure, wherein a decision as to the presence of a GW is made based on the test

$$\max_{t_0,\tau} \frac{\|\boldsymbol{r}\|^2/N}{\|\boldsymbol{r} - \widehat{\boldsymbol{s}}\|^2/N} \underset{H_0}{\overset{H_1}{\gtrless}} \Lambda_0 \tag{17}$$

which, based on (15), can be equivalently re-written, for sufficiently high N, as

$$\max_{t_0,\tau} \frac{Q_c\widehat{H}^2}{\frac{\|\boldsymbol{r}\|^2}{N}} = \max_{t_0,\tau} \frac{\frac{[<\boldsymbol{r},\boldsymbol{s}_c>]^2}{Q_c} + \frac{[<\boldsymbol{r},\boldsymbol{s}_s>]^2}{Q_c}}{\|\boldsymbol{r}\|^2/N} \underset{H_0}{\overset{H_1}{\gtrless}} \Lambda_0 \tag{18}$$

where $\widehat{H}^2 = \widehat{X}^2 + \widehat{Y}^2$.

Such a structure deserves some further comments. First, notice that the test statistic on the Left-Hand Side (LHS) does not depend on the noise PSD, which factors out the term on the LHS: as a consequence, the threshold level Λ_0 is itself independent of $\mathcal{N}_0/2$, implying that the receiver is non-parametric. Next, the numerator of (18), for increasingly high N, is just the squared envelope at the output of a filter matched to the (normalized) complex signal $[s_c(t) + js_s(t)]/Q_c$, in keeping with the results established in [2]. As to the denominator, it can be easily interpreted as an estimator of the noise PSD. In fact, $\|\boldsymbol{r}\|^2/N$ can be shown to converge in the mean square sense to $\mathcal{N}_0/2$ for increasingly high N under both H_1 and H_0 hypothesis. This is due to the fact that the useful signal admits only a limited number of significant projections along an orthonormal basis, so that the performance of the noise PSD estimator is not affected *asymptotically* by the hypothesis being actually in force: what is influenced by the presence of a useful signal in the received waveform is the *rate of convergence* of the estimator towards its limit.

So far, we have left aside the problem of maximizing the functional (18) with respect to τ and t_0. The maximization with respect to t_0 is easily accomplished by observing the signals at the output of a causal filter, matched to $[s_c(t) + js_s(t)]/Q_c$ and evaluating $\widehat{H}^2$ as the maximum value in the observation interval. Finding the maximum with respect to τ, would in principle require a bank of infinitely many filter, matched to the admissible signal durations: a viable solution is the classical "discrete bank of matched filters", wherein, upon discretization of the unknown parameter into L classes, the received signal is fed to a bank of receivers, each matched to one of the L admissible signal durations. Each receiver estimates the maximum $\widehat{H}^2$, and the maximum between the L selected values is finally adopted as an absolute maximum: a scheme of this detector is depicted in figure 1, wherein each block denoted as "receiver i" singles out the maximum $\widehat{H}^2$ which is observed in the interval $(0, T)$, assuming that the signal has duration τ_i.

3. NUMERICAL IMPLEMENTATION

So far the basis $\{\phi_i(t)\}$ has not been specified as yet, since, for increasingly high N, the final result is necessarily base-independent. If, conversely, the sub-optimum structure (17) is employed, the choice of the basis may turn out to be influential with respect to both the performance and the complexity of the detector. From now on, we adopt the following basis in $(0, T)$

$$\phi_k(t) = \begin{cases} \sqrt{\frac{1}{T}}, & k = 0 \\ \sqrt{\frac{2}{T}} \cos\left(\frac{\pi k}{T} t\right), & k \text{ even} \\ \sqrt{\frac{2}{T}} \sin\left[\frac{\pi(k+1)}{T} t\right], & k \text{ odd} \end{cases} \qquad (19)$$

Substituting back into (18), we have, for the terms in the numerator:

$$\widehat{X} = \frac{2}{Q_c T}\left[\mathcal{R}\left\{\sum_{k=1}^{(N-1)/2} R\left(\frac{k}{T}\right) S_c^*\left(\frac{k}{T}\right)\right\}\right] \qquad (20)$$

$$\widehat{Y} = -\frac{2}{Q_c T}\left[\mathcal{R}\left\{\sum_{k=1}^{(N-1)/2} R\left(\frac{k}{T}\right) S_s^*\left(\frac{k}{T}\right)\right\}\right] \qquad (21)$$

while, for the denominator, we have

$$\frac{\|r\|^2}{N} = \frac{1}{NT}\left[|R(0)|^2 + 2\sum_{k=1}^{(N-1)/2}\left|R\left(\frac{k}{T}\right)\right|^2\right] \qquad (22)$$

where $R(f) = \mathcal{F}[r(t)]$, $S_c(f) = \mathcal{F}[s_c(t)]$, and $S_s(f) = \mathcal{F}[s_s(t)]$ denote short-term (i.e., computed on the interval $(0, T)$) Fourier Transforms (FT), and the conditions $S_c(0) = S_s(0) = 0$ have been exploited.

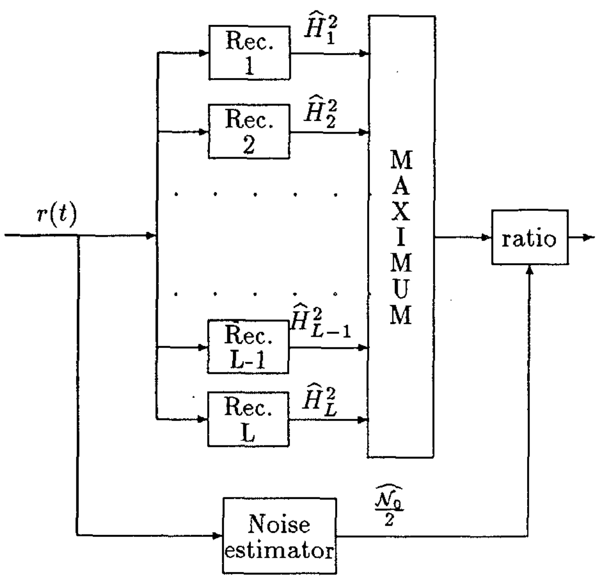

Figure 1: Non-parametric detector for GW

Even so, however, the receiver implementation requires evaluating and sampling the FT's of continuous-time waveforms. In order to show that the receiver admits a purely digital implementation, we first observe that feeding the received signal, which is bandwidth-unlimited under both hypothesis, to a pair of filters, matched to $s_c(t)$ and $s_s(t)$, respectively, is equivalent to feeding to the same filters a filtered version of $r(t)$, provided that the spectral content in the bandwidth of these filters remains practically unchanged. Thus, $r(t)$ can be pre-processed by an anti-aliasing filter whose bandwidth equals the common bandwidth, B say, of the two matched filters. Accordingly, if we denote by $r_B(t)$ the filtered version of $r(t)$, and by $T_c = 1/(2B)$ the sampling period, (21) are written, after some algebra:

$$\widehat{X} = \frac{T_c}{Q_c}\sum_{m=0}^{2L_B-1} r_B(mT_c)s_c(mT_c) \qquad (23)$$

$$\widehat{Y} = -\frac{T_c}{Q_c}\sum_{m=0}^{2L_B-1} r_B(mT_c)s_s(mT_c) \qquad (24)$$

wherein $L_B = BT$ and it has been implicitly assumed $N > 2L_B$.

Introducing the constraint that $s_c(t)$ and $s_s(t)$ be nonzero for $t_0 \leq t < t_0 + \tau$, we obtain

$$\widehat{X} = \frac{T_c}{Q_c}\left[\sum_{m=n_0}^{n_0+M_\tau-1} s_c(mT_c)r_B(mT_c)\right] \qquad (25)$$

$$\widehat{Y} = -\frac{T_c}{Q_c}\left[\sum_{m=n_0}^{n_0+M_\tau-1} s_s(mT_c)r_B(mT_c)\right] \qquad (26)$$

with $n_0 = t_0/T_c$ and $M_\tau = \tau/T_c$. We recall here that both n_0 and M_τ are *unknown* quantities, wherein numerical implementation of the optimized test still requires joint maximization with respect to these discrete parameters. Notice that, since (25, 26) represent cross correlations between the sampled versions of the received signal and $s_c(t)$ and $s_s(t)$, they can also be obtained by filtering the sequence $r_B(nT_c)$ through two filters, matched to $s_c(nT_c)$ and $s_s(nT_c)$, respectively. Precisely, if we introduce the sequences $\tilde{s_c}(n)$

and $\tilde{s}_s(n)$ as those obtained by sampling $s_c(t)$ and $s_s(t)$ as $t_0 = 0$, namely as two sequences lasting from 0 to $M_\tau - 1$, then the above correlations can be regarded as the output of the causal filters $h_c(n) = \tilde{s}_c(M_\tau - 1 - n)$ and $h_s(n) = \tilde{s}_c(M_\tau - 1 - n)$ at the time $n_0 + M_\tau - 1$: thus, searching for the maximum with respect to n_0 is tantamount to observing these outputs and selecting the largest one.

Unfortunately, a similar approach cannot be followed for the denominator of the test statistic (18), mainly because the received signal cannot be sampled without producing aliasing. However, we recall here that, unlike the useful signal, the noise occupies the entire receiver bandwidth, and that a reliable estimate of the noise floor could be achieved starting upon signal-free data. On the other hand, these data can obviously be collected by sampling the out-of-band part of the received signal, namely by considering the frequencies outside the useful bandwidth $(-B, B)$. Denote now by $W >> B$ the bandwidth of a low-pass filter, which introduces negligible distortion for $|f| < W$ and by $r_W(t)$ the received signal, as observed at the output of this filter. If $W \gg B$, an estimate of the noise PSD can be obtained from (22) with $R_W(f) = \mathcal{F}[r_W(t)]$ in place of $R(f)$. We stress here that the duration of this filtered version is approximately $T + 2/W \simeq T$; since, in our setup, $T >> 1/B$ and $1/B >> 1/W$, then the duration of the signal $r_W(t)$ is practically the same as $r(t)$.

Assuming $T_W = 1/(2W)$, it can be readily shown, based on the sampling theorem, that

$$R_W\left(\frac{k}{T}\right) \simeq T_W R_W(k)\ , \quad k < TW = L_W \tag{27}$$

wherein $R_W(k)$ denotes the Discrete FT (DFT), computed on $2L_W$ points, of the sequence $r_W(nT_W)$, whose length is approximately $2L_W$. With this approximations, we obtain

$$\frac{\widehat{\mathcal{N}_0}}{2} = \frac{T_W^2}{NT}\left[|R_W(0)|^2 + 2\sum_{k=1}^{L_W-1}|R_W(k)|^2\right] \tag{28}$$

which can be efficiently computed by resorting to FFT algorithms for calculating the DFT's. A scheme of the digital processing scheme implementing the optimized test is as depicted in figure 2, which substantially reproduces that of figure 1, except that the matched filter receivers operate on the sampled version of the received signal, according to equations (25,26) and the noise PSD estimation is performed via FFT. Precisely, the received signal is first fed to an anti-aliasing low-pass filter and subsequently sampled at frequency $2W$, thus forming the sequence $r_W(nT_W)$; the lower branch performs the estimate of the noise floor by averaging the square modula of the FFT of the received sequence, according to equation (28), so as to provide $\mathcal{N}_0/2$. In the upper branch, a numerical reduction of the sampling frequency is performed by first filtering the received sequence through a numerical low-pass filter whose cut-off frequency is $\nu_c = B/(2W)$, while the sequence $r_B(n)$ of the samples $r_B(nT_c) = r_B[n/(2B)]$ is extracted from the sequence $r_W(n)$ through decimation by a factor W/B (assumed heretoafter to be an integer). This sequence is forwarded, along with the estimated noise PSD, to the block denoted as "conventional receiver", inplementing the matched filterings and the search for the maxima with respect to n_0 and M_τ.

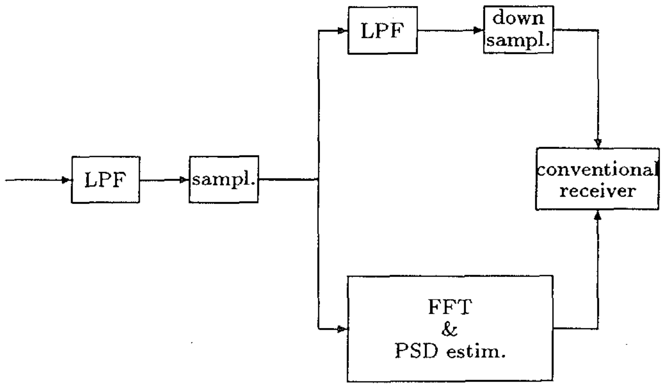

Figure 2: Numerical implementation of the detector

4. CONCLUDING REMARKS

In this paper the problem is considered of detecting gravitational waves from coalescing binaries in the presence of white noise with unknown power spectral density. Based on a Generalized Likelihood Ratio optimization approach, a new non-parametric detection structure has been presented, along with a possible digital implementation. Even though a thorough performance assessment has not been carried on as yet, preliminary results seem to indicate that this detector incurs a negligible detection loss, as measured with respect to the conventional receiver, operating in the presence of noise with known spectral characteristics. Even so, however, this detector is not able to combat all of the other noise components, i.e. seismic noise, whose spectral characteristics are only approximately known; a more realistic design setup should thus consider the presence of noise with either partially known or completely unknown covariance function: this topic is the object of current research.

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