SOUND FIELD GENERATED IN WATER LAYER OF CHANGING DEPTH BY A MOVING WATERBORNE SOURCE

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Abstract

The purpose of this paper is the construction and investigation of nonstationary normal waves excited in an ocean of changing depth by a moving source. In water, the sound speed depends on depth and weakly on the horizontal coordinates and time. The source is radiating a signal with varying amplitude and phase. The variation of different chcracteristics of the nonstationary normal waves in time is considered. It is shown that because of Doppler effect the new propagating nonstationary normal waves can arise. It results in noticeable change of the sound field of the moving source when compared with the sound field of the stationary source.

1. INTRODUCTION

At present in ocean acoustics, the nonstationary problems (in particular, the problems connected with a moving sound source) are very actual. The sound field generated in a water layer of constant depth by a moving waterborne source was investigated in [1]. But in many cases it is very important to consider the water layer of changing depth (a model for the continental slope).

The purpose of this paper is the construction and investigation of nonstationary normal waves excited in an ocean of changing depth by a moving waterborne source. In the water layer the sound speed depends on depth and weakly on the horizontal coordinates and time. The depth of the water layer weakly depends on the horizontal coordinates too. The bottom is assumed to be liquid. The source is radiating a signal with varying amplitude and phase. The unknowns are the nonstationary normal waves excited in the water layer. To find the nonstationary normal waves the nonstationary version of the method of horizontal rays/vertical modes is used (see [2], for example).

The typical feature of the present problem is the existence of a small parameter $\varepsilon \simeq 10^{-3} \div 10^{-2}$ characterizing the slow variation of the sound speed in water $c_W(z,x,y,t)$ on the horizontal variables and time. Besides, this parameter characterizes a weak dependence of the depth of the water layer $H(x,y)$ on the horizontal variables.

2. NONSTATIONARY NORMAL WAVES

The mathematical formulation of the considered problem is the following. Let $c_{\min} = \min c_W(z,x,y,t)$ be the minimum value of the sound speed in water and $d = c_{\min}/\max|\partial c_W/\partial z|$ be the characteristic spacial scale. Introduce dimensionless vertical variable $\zeta = z/d$, slow horizontal variables $\vec{\xi} = (\xi_1, \xi_2) = (\varepsilon x/d, \varepsilon y/d)$ and slow time $\tau = \varepsilon c_{\min} t/d$. In terms of dimensionless variables the equation of wave propagation with the right-hand side describing the moving point source is of the form

$$\frac{\partial^2 U}{\partial \zeta^2} + \varepsilon^2 \left[\frac{\partial^2 U}{\partial \xi_1^2} + \frac{\partial^2 U}{\partial \xi_2^2} - n^2(\zeta, \vec{\xi}, \tau) \frac{\partial^2 U}{\partial \tau^2} \right] = \\ = -A(\tau) \exp[i(q/\varepsilon)\varphi(\tau)] \delta(\zeta - \zeta_0(\tau)) \times \\ \times \ \delta(\vec{\xi} - \vec{\xi}_0(\tau)). \tag{1}$$

Here δ is the Dirac delta function; n is the index of refraction defined by the formula

$$n = \begin{cases} n_W = c_{\min}/c_W, \ 0 \le \zeta \le h, \\ n_H = c_{\min}/c_H, \ \zeta > h, \end{cases}$$

$h = h(\vec{\xi}) = H/d$ is the dimensionless depth of the ocean; c_H is the sound speed in a liquid bottom; $\zeta = \zeta_0(\tau)$, $\vec{\xi} = \vec{\xi}_0(\tau)$ is the trajectory of a source motion; $A(\tau)$ and $\varphi(\tau)$ are the amplitude and the phase of a source, respectively; $A(\tau)|_{\tau \le 0} = 0$. We shall assume that all the functions which are included into (1) have the derivatives of the order of unity and $\varphi'(\tau) < 0$, where a prime denotes the derivative with respect to slow time τ. The parameter q characterizes the value of instantaneous frequency of a source $\omega(t)$ in units of the characteristic frequency of the problem $c_{\min}/d$:

$$\omega(t) = -d[(q/\varepsilon)\varphi(\tau)]/dt = -q(c_{\min}/d)\varphi'(\tau) > 0.$$

At the air/water boundary $\zeta = 0$ and the bottom $\zeta = h(\vec{\xi})$, the following boundary conditions should be satisfied

$$U\Big|_{\zeta=0} = 0, \quad U\Big|_{\zeta=h-0} = U\Big|_{\zeta=h+0}, \\ \frac{\partial U}{\partial n}\Big|_{\zeta=h-0} \equiv \frac{\partial U}{\partial \zeta} - \varepsilon^2 (\nabla_\perp U, \nabla_\perp h)\Big|_{\zeta=h-0} = \nu \frac{\partial U}{\partial n}\Big|_{\zeta=h+0}. \tag{2}$$

Here ν is the ratio of the water density to the density of a liquid bottom, $\nabla_\perp \equiv \vec{i}\,\partial/\partial\xi_1 + \vec{j}\,\partial/\partial\xi_2$; $(\nabla_\perp U, \nabla_\perp h)$ is the scalar product of vectors $\nabla_\perp U$ and $\nabla_\perp h$.

The nonstationary normal waves U_m, $m = 1, 2, \ldots$ propagating in the water layer and excited by the moving waterborne source are seek in the form [1]

$$U_m = \exp[i(q/\varepsilon)\theta_m(\vec{\xi},\tau)] \sum_{j=0}^{\infty} \Psi_{mj}(\zeta,\vec{\xi},\tau)\,\varepsilon^j. \qquad (3)$$

The normal waves U_m should satisfy the homogeneous wave equation (1), the boundary conditions (2) as well as the principle of limiting absorption with $\zeta \to +\infty$. Moreover, U_m must satisfy the concordance conditions with the function describing the source with $\zeta - \zeta_0(\tau) \to 0$ and $|\vec{\xi} - \vec{\xi}_0(\tau)| \to 0$. In the expansion (3), phase function $\theta_m(\vec{\xi},\tau)$ and coefficients $\Psi_{mj}(\zeta,\vec{\xi},\tau)$, $m = 1, 2, \ldots$, $j = 0, 1, \ldots$ are to be determined.

Substituting U_m in the form (3) into the homogeneous wave equation (1) as well as into the boundary conditions (2) and equating coefficients of ε to various powers to zero we shall obtain a sequence of the Sturm-Liouville problems for the functions $\Psi_{mj}(\zeta,\vec{\xi},\tau)$ in the semi-infinite interval $0 \le \zeta < \infty$:

$$\begin{aligned}
\frac{d^2\Psi_{mj}}{d\zeta^2} &+ q^2\left[n^2\left(\frac{\partial\theta_m}{\partial\tau}\right)^2 - (\nabla_\perp\theta_m)^2\right]\Psi_{mj} + \\
&+ iq\Big\{2(\nabla_\perp\theta_m, \nabla_\perp\Psi_{m,j-1}) - \\
&- 2n^2\frac{\partial\theta_m}{\partial\tau}\frac{\partial\Psi_{m,j-1}}{\partial\tau} + \\
&+ \left[\Delta_\perp\theta_m - n^2\left(\frac{\partial^2\theta_m}{\partial\tau^2}\right)\right]\Psi_{m,j-1}\Big\} + \\
&+ \Delta_\perp\Psi_{m,j-2} - n^2\frac{\partial^2\Psi_{m,j-2}}{\partial\tau^2} = 0, \qquad (4)
\end{aligned}$$

$$\Psi_{mj}\Big|_{\zeta=0} = 0,\ \Psi_{mj} \underset{\zeta\to\infty}{\to} 0,\ \Psi_{mj}\Big|_{\zeta=h-0} = \Psi_{mj}\Big|_{\zeta=h+0} \qquad (5)$$

$$\begin{aligned}
\Big\{\frac{d\Psi_{mj}}{d\zeta} &- iq(\nabla_\perp\theta_m, \nabla_\perp h)\Psi_{m,j-1} - \\
&- (\nabla_\perp\Psi_{m,j-2}, \nabla_\perp h)\Big\}\Big|_{\zeta=h-0} = \\
&= \nu\Big\{\frac{d\Psi_{mj}}{d\zeta} - iq(\nabla_\perp\theta_m, \nabla_\perp h)\Psi_{m,j-1} - \\
&- (\nabla_\perp\Psi_{m,j-2}, \nabla_\perp h)\Big\}\Big|_{\zeta=h+0}. \qquad (6)
\end{aligned}$$

Here $\Delta_\perp = \nabla_\perp^2$, and we assume that $\Psi_{m,-2} = \Psi_{m,-1} = 0$.

For Ψ_{m0}, we get the homogeneous Sturm-Liouville problem. Let $\mu_m^2 = (\nabla_\perp\theta_m)^2$ be the eigenvalues and $f_m(\zeta) = f_m(\zeta,\vec{\xi},\tau)$ be the eigenfunctions of this problem. Obviously,

$$\Psi_{m0}(\zeta,\vec{\xi},\tau) = \mathcal{A}_{m0}(\vec{\xi},\tau) f_m(\zeta,\vec{\xi},\tau), \qquad (7)$$

where $\mathcal{A}_{m0} = \mathcal{A}_{m0}(\vec{\xi},\tau)$ is the function depending on $\vec{\xi}$ and τ only.

As μ_m depends on $\partial\theta_m/\partial\tau$, the equality $\mu_m^2 = (\nabla_\perp\theta_m)^2$ is the equation of the form

$$\mu_m^2(\partial\theta_m/\partial\tau) = (\nabla_\perp\theta_m)^2 \qquad (8)$$

for the phase function $\theta_m(\vec{\xi},\tau)$ of the nonstationary normal wave. The dependence of μ_m on the derivative $\partial\theta_m/\partial\tau$ essentially distinguishes the nonstationary eikonal equation (8) from the classical eikonal equation for the normal wave in the method of horizontal rays/vertical modes [2].

It is possible to rewrite the nonstationary eikonal equation (8) in the form solved for $\partial\theta_m/\partial\tau$:

$$\partial\theta_m/\partial\tau + \mathcal{H}_m((\nabla_\perp\theta_m)^2, \vec{\xi}, \tau) = 0. \qquad (9)$$

Here

$$\mathcal{H}_m \equiv \left[(\nabla_\perp\theta_m)^2 N_m^2 M_m^{-2} + q^{-2}(N_m')^2 M_m^{-2}\right]^{1/2}$$

and the following notations are introduced

$$\begin{aligned}
N_m^2 &= \left(\int_0^h + \nu\int_h^\infty\right) f_m^2(\zeta)\,d\zeta, \\
M_m^2 &= \left(\int_0^h + \nu\int_h^\infty\right) n^2 f_m^2(d\zeta)\,d\zeta, \\
(N_m')^2 &= \left(\int_0^h + \nu\int_h^\infty\right) (df_m/d\zeta)^2 d\zeta.
\end{aligned}$$

It can be shown that the function $\mathcal{H}_m$ is independent of $\partial\theta_m/\partial\tau$ despite the fact that the eigenfunction $f_m(\zeta)$ and consequently the integrals $N_m^2, M_m^2, (N_m')^2$ depend on $\partial\theta_m/\partial\tau$.

The Hamilton-Jacobi equation (9) should be supplemented with the initial data connecting the function $\theta_m(\vec{\xi},\tau)$ with the phase function of the source at the trajectory of its motion. If the trajectory of the source movement is given in parametric representation $\tau = \alpha$, $\vec{\xi} = \vec{\xi}_0(\alpha)$, the equality of the phase functions at the source can be expressed as

$$\theta_m(\vec{\xi},\tau)\Big|_{\tau=\alpha,\,\vec{\xi}=\vec{\xi}_0(\alpha)} = \varphi(\alpha). \qquad (10)$$

As Eq. (9) is nonlinear it is necessary to give the values of derivatives $\partial\theta_m/\partial\tau$ and $\partial\theta_m/\partial\vec{\xi}$ as well:

$$\begin{aligned}
\partial\theta_m/\partial\tau\Big|_{\tau=\alpha,\,\vec{\xi}=\vec{\xi}_0(\alpha)} &= -\Omega_m, \\
\partial\theta_m/\partial\vec{\xi}\Big|_{\tau=\alpha,\,\vec{\xi}=\vec{\xi}_0(\alpha)} &= K_m\vec{e},
\end{aligned} \qquad (11)$$

where $\vec{e} = (\cos\vartheta, \sin\vartheta)$ is the unit vector. The initial data on the right-hand sides of (11) must satisfy the concordance conditions

$$\begin{aligned}
\Omega_m &= \mathcal{H}_m(K_m^2, \vec{\xi}_0(\alpha), \alpha), \\
\Omega_m &= K_m(\vec{e}, \vec{v}(\alpha)) - \varphi'(\alpha).
\end{aligned} \qquad (12)$$

Here $\vec{v}(\alpha) = d\vec{\xi}_0(\tau)/d\tau|_{\tau=\alpha}$ is the dimensionless velocity of a moving source at instant $\tau = \alpha$. The equalities (11) define Ω_m and K_m as functions of α and ϑ.

The solutions of the Hamilton-Jacobi equation (9) satisfying the initial conditions (10), (11) can be found by means of the method of characteristics [3]. The characteristic system corresponding to (9) is written in the form

$$\frac{d\tau}{ds}=1,\ \frac{d\xi_i}{ds}=\frac{\partial\mathcal{H}_m}{\partial p_i},\ \frac{dp_0}{ds}=-\frac{\partial\mathcal{H}_m}{\partial\tau},\quad \frac{dp_i}{ds}=-\frac{\partial\mathcal{H}_m}{\partial\xi_i},\ i=1,2\ , \tag{13}$$

where $\mathcal{H}_m=\mathcal{H}_m(p^2,\vec{\xi},\tau)$, $p=|\vec{p}|$. The initial data are given as follows

$$\tau\Big|_{s=0}=\alpha,\ \ \vec{\xi}\Big|_{s=0}=\vec{\xi}_0(\alpha),\quad p_0\Big|_{s=0}=-\Omega_m,\ \ \vec{p}\Big|_{s=0}=K_m\vec{e}. \tag{14}$$

If the solutions of the characteristic system (13) satysfying the initial conditions (14) are known

$$\tau=s+\alpha,\ \ \xi_i(s)=\xi_i(\alpha,\vartheta,s),\quad p_0(s)=p_0(\alpha,\vartheta,s),\ p_i(s)=p_i(\alpha,\vartheta,s),\ i=1,2, \tag{15}$$

the phase function $\theta_m(\vec{\xi},\tau)$ is found according to the method of characteristics

$$\theta_m(\vec{\xi},\tau)=\varphi(\alpha)+\int_0^s\left[p_o(s)+\sum_{i=1}^{2}p_i(s)\,d\xi_i(s)/ds\right]ds,$$

where the integral is taken along the characteristic.

The parameters α,ϑ,s (the space-time ray coordinates) are connected with the Cartesian coordinates ξ_1,ξ_2 and time τ by the first three equations of (15). Under fixed α and ϑ these equations give the parametric representation of a space-time ray.

The ray coordinates have the following meaning: the coordinate α is the instant of radiation of the field of a normal wave registered at the point $\vec{\xi}$ at the moment τ; ϑ is the angle of the ξ_1-axis with the horizontal ray emerging from the source at the instant of radiation and arriving to the observation point (ξ_1,ξ_2); s is the time of propagation of the normal wave field.

Let us obtain the equation for the factor $\mathcal{A}_{m0}(\vec{\xi},\tau)$ in the expression (7) for Ψ_{m0}. This factor must be chosen so that the nonhomogeneous Sturm-Liouville problem (4)–(6), $j=1$ for Ψ_{m1} be solvable. Introducing the three-dimensional vector $\vec{W}_m=(V_{m,1},V_{m,2},1)$, where $\vec{V}_m=(V_{m,1},V_{m,2})=\partial\mathcal{H}_m/\partial(\nabla_\perp\theta_m)$, the solvability condition can be written in the divergence form

$$\operatorname{div}(\mathcal{E}_m\vec{W}_m)+\gamma_m\mathcal{E}_m=0. \tag{16}$$

Here the following notations are introduced

$$\gamma_m=(1-\nu)(\vec{V}_m,\nabla_\perp h)f_m^2(h)N_m^{-2}-L_m^2M_m^{-2},$$
$$\mathcal{E}_m=-(\partial\theta_m/\partial\tau)M_m^2\mathcal{A}_{m0}^2,$$
$$L_m^2=\int_0^h(\partial n_W^2/\partial\tau)f_m^2(\zeta)\,d\zeta.$$

Transforming $\operatorname{div}(\mathcal{E}_m\vec{W}_m)$ in the usual way to the form $J_m^{-1}\partial(J_m\mathcal{E}_m)/\partial s$, where J_m is the Jacobian of transformation from the Cartesian coordinates ξ_1,ξ_2 and time τ to the ray coordinates, we obtain the new form of the transfer equation (16) — as an ordinary differential equation along a space-time ray. Integrating this equation, we get

$$\mathcal{A}_{m0}=\mathcal{B}_{m0}(\alpha,\vartheta)[-(\partial\theta_m/\partial\tau)M_m^2J_m]^{-1/2}\times\ \times\ \exp\left(-\frac{1}{2}\int_0^s\gamma_m ds\right).$$

The multiplier $\mathcal{B}_{m0}(\alpha,\vartheta)$ (the excitation coefficient of the normal wave in the adiabatic approximation) is found with the help of the principle of localization from the comparison of the main term of the expansion (3) with the nonstationary normal waves in the exactly solvable problem for the stratified water layer with $H=\text{const}$ and the waterborne source moving rectilinearly and uniformly at the fixed depth [4]. The formula for $\mathcal{B}_{m0}$ has the form

$$\mathcal{B}_{m0}(\alpha,\vartheta)=e^{i\frac{\pi}{4}}A(\alpha)[8\pi(q/\varepsilon)]^{-1/2}\times\ \times\ (1-\beta_m\cos\vartheta)^{-1/2}\mathring{f}_m(\zeta_0(\alpha)).$$

Here $A(\alpha)$ is the amplitude of a source at the instant α of radiation of the field of a normal wave; $\beta_m=v/v_m^G$; v is the constant dimensionless velocity of a model source; $v_m^G=|\vec{v}_m^G|$; $\vec{v}_m^G=\partial\mathcal{H}_m/\partial\vec{p}$ is the group velocity vector of the nonstationary normal wave; $\mathring{f}_m(\zeta)=f_m(\zeta)(N_m^2)^{-1/2}$ is the normalized eigenfunction.

The analysis of the system including the first three equations of (15) as well as the equations (12) demonstrates that in a waveguide because of Doppler effect the new propagating nonstationary normal waves can arise. If results in noticeable change of the sound field of the moving source when compared with the sound field of the stationary source.

References

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