Time-frequency representations of generalized almost-cyclostationary signals

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RÉSUMÉ

Nous proposons dans cet article une contribution à la théorie récemment introduite des signaux quasi-cyclostationnaires généralisés. Cette classe de signaux étend la classe des signaux quasi-cyclostationnaires au cas de signaux dont les fréquences cycliques dépendent du temps. Des représentations temps-fréquence de ces signaux sont données en fonction des statistiques cycliques généralisées. Notamment, la distribution de Wigner-Ville et la fonction d'ambiguité sont examinées en détail. Le problème de l'extraction des caractéristiques des signaux quasi-cyclostationnaires généralisés, basée sur l'estimation d'un seul enregistrement, est également traité.

ABSTRACT

A contribution to the theory of very recently introduced generalized almost-cyclostationary signals is given. Such a class of signals generalizes that of the almost-cyclostationary signals to the case in which the cycle frequencies depend on the lag parameter. Time-frequency representations for these signals are expressed in terms of generalized cyclic statistics. The Wigner-Ville distribution and the ambiguity function are examined in detail. For generalized almost-cyclostationary signals, the problem of signal feature extraction based on single-record estimation is addressed.

1 Introduction

Very recently, the class of higher-order generalized almost-cyclostationary (GACS) time-series has been introduced [5], [6]. Time series belonging to this class are characterized by multivariate statistical functions that are almost-periodic functions of time whose Fourier series expansions can exhibit coefficients and frequencies depending on the lag shifts of the time series. Moreover, the union over all the lag shifts of the lag-dependent frequency sets is not necessarily countable. Almost-cyclostationary (ACS) time-series turn out to be the subclass of GACS time-series for which the frequencies do not depend on the lag shifts and the union of the above mentioned sets is countable. Examples of GACS time-series not belonging to the subclass of ACS time-series arise from some linear time-variant transformations of ACS time-series, such as channels introducing a time-varying delay [6]. Chirp signals [5] and several time-jittered communication signals are further examples. For those GACS signals that are not ACS the second-order wide-sense characterization in terms of cyclic autocorrelation functions and cyclic spectra is inadequate. Then, lag-dependent cycle frequencies, generalized cyclic autocorrelation functions and generalized cyclic spectra must be introduced.

This paper deals with time-frequency representations of second-order GACS signals. Specifically, after a brief introduction (Section 2) on GACS signals, in Section 3 the Cohen's general class of time-frequency distributions [2] of GACS signals is considered and it is shown that any representation belonging to this class is expressed as sum of two terms. The first one involves the generalized cyclic statistics of the signal; the second one is related to the residual term obtained by subtracting to the second-order lag product its almost-periodic component (the time-varying autocorrelation function). The Wigner-Ville distribution is examined in detail. Moreover, the ambiguity function is considered. It is shown that it can be expressed as sum of impulsive terms related to the generalized cyclic statistics of the signal and a nonimpulsive component related to the above mentioned residual term. Furthermore, the subclass of ACS signals is examined. In Section 4, the problem of signal feature extraction is considered. It is shown that the estimation of the cyclic autocorrelation as a function of cycle frequency and lag parameter allows one to determine the lag-dependent cycle frequencies and the generalized cyclic autocorrelation functions annihilating the effect of the residual term when the collect time increases. Then, an estimate of the time-varying autocorrelation function can be easily derived. On the contrary, in general, in time-frequency representations the component related to the residual term cannot be separated by the components related to the generalized cyclic statistics.

Finally, let us note that the time-frequency distributions and the ambiguity function were originally defined with reference to finite-energy signals. Moreover, finite-power signals can be considered by employing Dirac's delta functions (see, e.g., sine waves and chirp signals in [2]). The approach adopted in this paper follows this line since GACS time-series exhibit finite power. However, it is worthwhile to underline that a different approach is adopted in [3] where for a time-windowed ACS signal the Wigner-Ville distribution is related to the cyclic periodograms and the ambiguity function is recognized to be equal, but for a scale factor, to the cyclic correlogram.

2 Second-order GACS signals

In the fraction-of-time probability framework, a continuous-time finite-power time-series $x(t)$ is said to exhibit second-order wide-sense cyclostationarity with cycle frequency $\alpha \neq 0$ if the cyclic autocorrelation function

$$R_x^\alpha(\tau) \triangleq \left\langle x(t+\tau/2)x^*(t-\tau/2)\, e^{-j2\pi\alpha t}\right\rangle \tag{1}$$

exists and is not zero for some τ [3]. In (1), $*$ represents complex conjugation and $\langle\cdot\rangle$ denotes infinite-time averaging. Analogously, the time series is said to exhibit second-order wide-sense conjugate cyclostationarity if the conjugate cyclic autocorrelation function

$$R_{xx^*}^\alpha(\tau) \triangleq \left\langle x(t+\tau/2)x(t-\tau/2)\, e^{-j2\pi\alpha t}\right\rangle$$

exists and is not zero for some τ. In the following, we will deal with time series exhibiting cyclostationarity. The consideration of time series exhibiting conjugate cyclostationarity will require some obvious minor changes.

If the set

$$A_\tau \triangleq \left\{\alpha \in \mathrm{R} \;:\; R_x^\alpha(\tau) \neq 0\right\}$$

is countable for each τ, then the time series is said to be second-order generalized almost-cyclostationary in the wide-sense [5] and the almost-periodic function

$$\begin{aligned} R_x(t,\tau) \;&\triangleq\; E^{\{A_\tau\}}\left\{x(t+\tau/2)x^*(t-\tau/2)\right\} \\ &=\; \sum_{\alpha\in A_\tau} R_x^\alpha(\tau)\, e^{j2\pi\alpha t}, \end{aligned} \tag{2}$$

where $E^{\{A_\tau\}}\{\cdot\}$ denotes the almost-periodic component extraction operator [3], is referred to as the time-varying autocorrelation function. Then, the lag product $x(t+\tau/2)x^*(t-\tau/2)$ can be expressed as the sum of its almost-periodic component and a residual term not containing additive sine wave components:

$$x(t+\tau/2)x^*(t-\tau/2) \triangleq R_x(t,\tau) + \ell_x(t,\tau)\,, \tag{3}$$

where

$$\left\langle \ell_x(t,\tau)\, e^{-j2\pi\alpha t}\right\rangle \equiv 0, \qquad \forall\alpha \in \mathrm{R}. \tag{4}$$

In the case where the set

$$A \triangleq \bigcup_{\tau\in\mathrm{R}} A_\tau$$

is countable, the time-series $x(t)$ is said to be second-order wide-sense ACS.

Almost-cyclostationary time-series can be characterized in the frequency domain by the cyclic spectra $S_x^\alpha(f)$ which are the Fourier transforms of the corresponding cyclic autocorrelation functions. However, as shown in [5], such a characterization is not appropriate for those GACS time-series that are not ACS, that is, when the set A is not countable. In fact, in such a case, even if the set A_τ and the time-varying autocorrelation function $R_x(t,\tau)$ are continuous functions of τ, the cyclic autocorrelation functions are not necessarily continuous functions of τ and then the cyclic spectra can result to be infinitesimal.

A useful characterization in the frequency domain can be introduced for those GACS time-series for which the set A_τ and the function $R_x(t,\tau)$ are both continuous with respect to τ. In fact, in such a case the support in the (α,τ) plane of the cyclic autocorrelation function can be written as

$$\operatorname{supp}\left\{R_x^\alpha(\tau)\right\} \triangleq \left\{(\alpha,\tau)\in A_\tau\times\mathrm{R} \;:\; R_x^\alpha(\tau)\neq 0\right\}$$

$$= \bigcup_{\zeta\in W}\left\{(\alpha,\tau)\in\mathrm{R}\times\mathrm{R} \;:\; \alpha=\alpha_\zeta(\tau),\; R_x^\alpha(\tau)\neq 0\right\},$$

where W is a countable set and the (reduced-dimension) lag-dependent cycle frequencies $\alpha_\zeta(\tau)$ are continuous functions of τ. Therefore, if one further assumes that for $\zeta' \neq \zeta$ it results that

$$\alpha_{\zeta'}(\tau) \neq \alpha_\zeta(\tau), \qquad \tau\in\mathrm{R}-\mathrm{D},$$

where D is a set with zero measure in R that does not contain cluster points, then the time-varying autocorrelation function (2) can be expressed as

$$R_x(t,\tau) = \sum_{\zeta\in W} R_{x,\zeta}(\tau)\, e^{j2\pi\alpha_\zeta(\tau)t}, \tag{5}$$

where the functions

$$R_{x,\zeta}(\tau) \triangleq \lim_{\Delta\tau\to 0}\Big\langle x(t+(\tau+\Delta\tau)/2)$$

$$\cdot x^*(t-(\tau+\Delta\tau)/2)\, e^{-j2\pi\alpha_\zeta(\tau+\Delta\tau)t}\Big\rangle, \tag{6}$$

referred to as the generalized cyclic autocorrelation functions, turn out to be continuous functions also when the cyclic autocorrelation functions are not. It is useful to point out that the limit for $\Delta\tau\to 0$ is introduced into definition (6) to avoid discontinuities in $R_{x,\zeta}(\tau)$ in correspondence of those $\tau\in\mathrm{D}$ such that, for some $\zeta'\neq\zeta$, one has $\alpha_{\zeta'}(\tau)=\alpha_\zeta(\tau)$.

It can be easily shown that the cyclic autocorrelation functions and the generalized cyclic autocorrelation functions are related by the following relationships:

$$R_{x,\zeta}(\tau) = \lim_{\Delta\tau\to 0} R_x^\alpha(\tau+\Delta\tau)\Big|_{\alpha=\alpha_\zeta(\tau+\Delta\tau)},$$

$$R_x^\alpha(\tau) = \sum_{\zeta\in W} R_{x,\zeta}(\tau)\delta_{\alpha-\alpha_\zeta(\tau)},$$

where $\delta_\gamma = 1$ for $\gamma = 0$ and $\delta_\gamma = 0$ for $\gamma \neq 0$.

Let us note that for the ACS time-series the functions $\alpha_\zeta(\tau)$ are independent of τ and then there exists a one-to-one correspondence between the elements ζ of the set W and the cycle frequencies α belonging to the set A. Moreover, for each α and ζ such that $\alpha_\zeta(\tau)=\alpha$, it results that

$$R_{x,\zeta}(\tau) = R_x^\alpha(\tau). \tag{7}$$

The Fourier transform of the generalized cyclic autocorrelation function

$$S_{x,\zeta}(f) \triangleq \int_{\mathrm{R}} R_{x,\zeta}(\tau)\, e^{-j2\pi f\tau}\,\mathrm{d}\tau \tag{8}$$

is called the generalized cyclic spectrum. In the special case of ACS time-series, it is coincident with the cyclic spectrum.

3 Time-frequency representations of GACS signals

All time-frequency distributions for a complex-valued time-series $x(t)$ can be obtained from

$$C_x(t, f) = \int_{\mathrm{R}^3} x(u + \tau/2)x^*(u - \tau/2)$$

$$\cdot\phi(\theta, \tau)\, e^{-j2\pi\theta(t-u)}\, e^{-j2\pi f\tau}\, \mathrm{d}u\, \mathrm{d}\tau\, \mathrm{d}\theta$$

$$= \int_{\mathrm{R}} x(t + \tau/2)x^*(t - \tau/2) \underset{t}{\otimes} \Phi(t, \tau)\, e^{-j2\pi f\tau}\, \mathrm{d}\tau, \qquad (9)$$

where

$$\Phi(t, \tau) \triangleq \int_{\mathrm{R}} \phi(\theta, \tau)\, e^{-j2\pi\theta t}\, \mathrm{d}\theta$$

and $\underset{t}{\otimes}$ denotes convolution with respect to t. The kernel function $\phi(\theta, \tau)$ determines the distribution and its properties [2].

By substituting (3) into (9) and accounting for (5) and (8), the expression of the generic time-frequency distribution in terms of generalized cyclic statistics can be obtained:

$$C_x(t, f) = \sum_{\zeta\in W} S_{x,\zeta}(f) \underset{f}{\otimes} A_\zeta^\Phi(t, f) + \mathscr{L}_x^\Phi(t, f)\,, \qquad (10)$$

where

$$A_\zeta^\Phi(t, f) \triangleq \int_{\mathrm{R}} e^{j2\pi\alpha_\zeta(\tau)t} \underset{t}{\otimes} \Phi(t, \tau)\, e^{-j2\pi f\tau}\, \mathrm{d}\tau \qquad (11)$$

and

$$\mathscr{L}_x^\Phi(t, f) \triangleq \int_{\mathrm{R}} \ell_x(t, \tau) \underset{t}{\otimes} \Phi(t, \tau)\, e^{-j2\pi f\tau}\, \mathrm{d}\tau. \qquad (12)$$

In other words, all time-frequency distributions of GACS time-series can be expressed as the sum of two contributions. The first one is the sum of all generalized cyclic spectra each spreaded (in the frequency domain) by the time-varying function $A_\zeta^\Phi(t, f)$ depending on the corresponding (reduced-dimension) lag-dependent cycle frequency $\alpha_\zeta(\tau)$ and the kernel function. The second one is related to the residual term $\ell_x(t, \tau)$.

By adopting the kernel function $\phi(\theta, \tau) = 1$ in (9), one obtains the Wigner-Ville distribution [2]

$$W_x(t, f) \triangleq \int_{\mathrm{R}} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi f\tau}\, \mathrm{d}\tau,$$

which, accounting for (10)-(12), can be expressed as

$$W_x(t, f) = \sum_{\zeta\in W} S_{x,\zeta}(f) \underset{f}{\otimes} \underset{\tau\to f}{\mathscr{F}}\left\{e^{j2\pi\alpha_\zeta(\tau)t}\right\} + \underset{\tau\to f}{\mathscr{F}}\left\{\ell_x(t, \tau)\right\}\,,$$

where $\underset{\tau\to f}{\mathscr{F}}$ denotes the Fourier transform operator from the τ domain to the f domain.

In the special case of ACS time-series the (reduced dimension) lag-dependent cycle frequencies are constant and, hence, (10) reduces to

$$C_x(t, f) = \sum_{\alpha\in A} S_x^\alpha(f) \underset{f}{\otimes} \Psi(\alpha, f)\, e^{j2\pi\alpha t} + \mathscr{L}_x^\Phi(t, f)\,, \qquad (13)$$

where

$$\Psi(\alpha, f) \triangleq \int_{\mathrm{R}^2} \Phi(t, \tau)\, e^{-j2\pi(\alpha t + f\tau)}\, \mathrm{d}t\, \mathrm{d}\tau\,,$$

and (7) has been accounted for. Moreover, for the Wigner-Ville distribution $\Psi(\alpha, f) = \delta(f)$ and, hence,

$$W_x(t, f) = \sum_{\alpha\in A} S_x^\alpha(f)\, e^{j2\pi\alpha t} + \underset{\tau\to f}{\mathscr{F}}\left\{\ell_x(t, \tau)\right\}\,, \qquad (14)$$

which is just the result derived in [4] except for the component depending on $\ell_x(t, \tau)$. The absence of such a term in [4] stems from the fact that for ACS signals, in the stochastic process framework (adopted in [4]), $\ell_x(t, \tau)$ is dropped out by the statistical expectation operation. However, let us note that such a residual term is present also in the stochastic process approach when asymptotically mean ACS (AMACS) processes [1] are considered. Moreover, it is worthwhile to underline that the residual term is always present in the single-record based estimate of the Wigner-Ville distribution that, for both ACS and AMACS processes, asymptotically approaches expression (14) when the collect time increases. Multiple-record estimates of the Wigner-Ville distribution lead to a zero residual term. However, they can be singled out only when the signal is cyclostationary (i.e., all the cycle frequencies are multiple of a fundamental one) and the period of cyclostationarity is known [7].

The ambiguity function

$$A_x(\nu, \tau) \triangleq \int_{\mathrm{R}} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi\nu t}\, \mathrm{d}t\,,$$

for GACS time-series, accounting for (3) and (5), can be expressed in terms of generalized cyclic statistics:

$$A_x(\nu, \tau) = \sum_{\zeta\in W} R_{x,\zeta}(\tau)\, \delta(\nu - \alpha_\zeta(\tau)) + \underset{t\to\nu}{\mathscr{F}}\left\{\ell_x(t, \tau)\right\}\,, \qquad (15)$$

where $\delta(\cdot)$ denotes Dirac's delta function. Equation (15) shows that the ambiguity function of GACS signals is the sum of some impulsive terms whose supports are curves described by the lag-dependent cycle frequencies and whose amplitudes are the generalized cyclic autocorrelation functions and an aperiodic component that, accounting for (4), does not contain impulses.

Finally, let us note that, in the special case of ACS time-series, (15) specializes to

$$A_x(\nu, \tau) = \sum_{\alpha\in A} R_x^\alpha(\tau)\, \delta(\nu - \alpha) + \underset{t\to\nu}{\mathscr{F}}\left\{\ell_x(t, \tau)\right\}\,.$$

4 Signal feature extraction

In problems of signal feature extraction for GACS time-series, in general, no a priori knowledge exists on the possible cyclostationary nature of the signal. Therefore, single-record estimators of time-frequency distributions and generalized cyclic statistics must be utilized. The estimators are obtained directly by applying the definitions where, however, integrals and time averages are performed over a finite collect-time. Then, they asymptotically approach the theoretical values when the observation time increases. Once the lag-dependent cycle frequencies and/or the generalized cyclic statistics have been estimated, the time-varying autocorrelation function can be reconstructed, signal parameters can be estimated, and signals can be classified on the basis of their different generalized cyclic statistic characteristics.

It is worthwhile to underline that, in general, in time-frequency representations of GACS time-series, the component related to the residual term $\ell_x(t,\tau)$ cannot be separated from the component related to the cyclic statistics (see (10)). In the special case of ACS time-series, however, from (13) it follows that the component related to the cyclic statistics is almost periodic and, hence, algorithms for estimating amplitude and frequencies of almost-periodic signals embedded in noise can be exploited to obtain estimates of the cyclic parameters of interest.

The role played by the residual term can be illustrated by an example. Specifically, let us consider the signal

$$x(t) = \int_{\mathrm{R}} h(t,u) s(u)\,\mathrm{d}u,$$

where

$$s(t) = w(t)\ \exp(j2\pi f_0 t)$$

and

$$h(t,u) = \delta(u - t + D(t))$$

is the impulse-response function of a channel introducing a time-varying delay $D(t)$. Figure 1 shows the magnitude of the cyclic autocorrelation function $R_x^\alpha(\tau)$, for the signal $x(t)$, as a function of α and τ, as estimated by 256 samples. It has been assumed that $f_0 = 0.04/T_s$, where T_s is the sampling period, and the real signal $w(t)$ is wide-sense stationary with power spectral density $S_w^0(f) = (1 + f^2/B^2)^{-8}$ with $B = 0.015/T_s$. Moreover, a time-varying delay $D(t) = d_1 t + d_2 t^2$ with $d_1 = 0.25$ and $d_2 = 0.02/T_s$ has been considered. The support of $R_x^\alpha(\tau)$ is constituted by curves described by the lag-dependent cycle frequencies $\alpha_\zeta(\tau)$; on each of them, the cyclic autocorrelation function is just equal to the corresponding generalized cyclic autocorrelation function $R_{x,\zeta}(\tau)$. Figure 2 shows the magnitude of the Wigner-Ville distribution for the same signal. The presence of a component related to the residual term is evident.

Finally, let us observe that the estimates of the cyclic autocorrelation function and the ambiguity function differ only for a scaling factor [3]. Therefore, the above cyclic autocorrelation function based estimation procedure can also be interpreted in terms of ambiguity function.

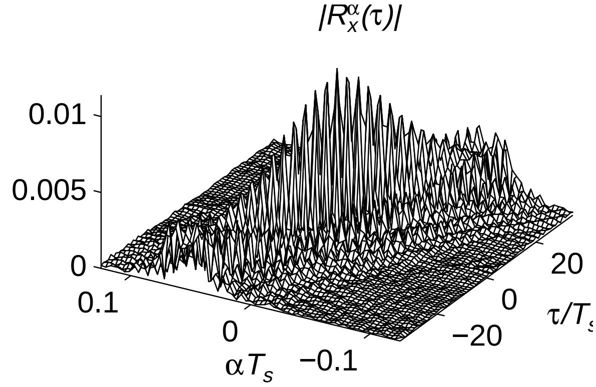

Figure 1.

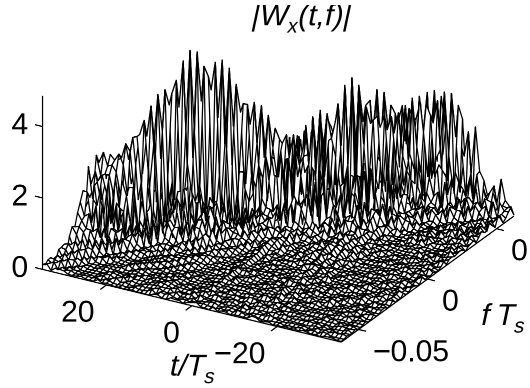

Figure 2.

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