Local Dynamical Analysis of the Invariant Set of a Signal

Karin Vibe-Rheymer, Jean-Marc Vesin

Laboratoire de Traitement des Signaux

Ecole Polytechnique Fédérale de Lausanne

LTS/DE - EPFL

CH- 1015 Lausanne, Suisse

RÉSUMÉ

Bien que l'analyse globale des systèmes nonlinéaires soit beaucoup utilisée, peu d'essais ont été faits pour tenter de charactériser leur comportement local, et les succès obtenus furent limités. Ceci provient principalement de ce que ces tentatives se sont concentrées sur l'estimation quantitative du comportement, et que les zones d'intérêt ont été déterminées de manière assez arbitraire. Cet article se propose de résoudre le problème en présentant une technique estimant l'exposant de Lyapunov local pour tous les points de l'attracteur et les agglomérant en groupes significatifs. Les exemples basés sur le système de Lorenz montrent que, si les résultats sont souvent peu fiables numériquement, ils permettent une analyse qualitative de l'attracteur, et les groupes formés ont une signification physique.

ABSTRACT

While the global analysis of nonlinear dynamical systems is of wide use, it characterizes only the global behavior of a system. Attempts have been made to estimate local measures, hence local behaviors, but these have not been very sucessful, mainly because they focused on a quantitative analysis, and divided the attractor into regions of interest rather arbitrarily. This paper presents a technique for computing estimates of the local Lyapunov exponent on each point of the attractor, then group them into meaningful clusters. Examples based on the Lorenz system will show that, while the results do not yield accurate estimates, a qualitative analysis of the attractor remains possible. Moreover, the clusters obtained can be linked to physical behaviors.

1   Introduction

Recent works in the literature have shown the interest of computing local measures on the attractor of chaotic signals. Global measures, such as attractor dimension or the Lyapunov spectrum, can indeed indicate global features of the system, but do not tell about its local behavior. It has however been shown that local Lyapunov exponents could be very useful to determine the local predictability of a signal [1, 2].

These works divide the attractor into regions, for which a center is defined. It is then possible to compute the local Lyapunov spectrum, using the Jacobian matrix of the system [2]. An alternative way is to compute the local divergence rate, from which the local largest Lyapunov exponent can be computed [3, 4, 5].

These techniques have not been much used, for two main reasons : because of the limited amount of points in each region, the estimate of the local exponent is often inaccurate. Moreover, the regions of interest are chosen in an arbitrary fashion, and may not correspond to the natural boundaries between different value ranges of the local Lyapunov exponents. Finally, the choice of these regions may hide smaller regions between which the exponent varies significantly, thus hiding part of the local behavior.

This paper tries to partially solve these problems by proposing a technique associating a measure to each data point, before grouping nearby measures of similar values on the attractor. The measure of choice is the largest local Lyapunov exponent, but other measures, such as local spatial dimension,

can be used. The next section describes how the local exponent is computed, then the way the clusters are made. Results are given, using the Lorenz system as an example.

2   Theory

2.1   Local Lyapunov exponent computation

There are two main ways of computing the Lyapunov exponents. The first one uses an estimate of the Jacobian matrix of the system [6]. It basically consists in estimating a derivative of the system map, then find its eigenvalues, which yield the Lyapunov exponents. The problem of this method is that it is very sensitive to noise, which alters the eigenvalues. This can result in spurious Lyapunov exponents, and a wrong estimate of the real ones.

The number of data available for the estimation of the Jacobian matrix is also critical. It was proposed to solve this problem by using multiple realizations of the system having different initial conditions [1]. This however requires very specific conditions : Either the physical system itself must be available, so that experiments can be done, or the equations modeling its behavior must be known, so that simulations can be run. These conditions are often not met in practice, for which the typical case is a single realization of the system, with no or scarce knowledge of its behavior.

It is therefore necessary to use the other Lyapunov exponent estimation method, that uses a numerical approach. First proposed by Wolf in [7], it consists in estimating the divergence

rate between two nearby trajectories according to time, by monitoring the evolution of close elements of the attractor. While theoretically allowing the estimation of all exponents, it is used only for the computation of the largest Laypunov exponent. This technique has shown its usefulness and its efficiency : in particular, its most advanced implementations [8, 9] are very robust to noise and length of the data [10].

The chosen technique is based on an older version of the numerical approach. In [3], Wolff proposed a computation scheme for the local Lyapunov exponents, along with the theoretical background justifying the idea of local exponents. It is however limited to one-dimensional systems and needs be extended to multi-dimensional ones.

Let it be a time series $x(n) = [x_1, x_2, \cdots, x_N]$. Its divergence rate is directly proportional to the Lyapunov exponent, and will be positive if the system having generated this time series is chaotic. If the underlying system governing the signal is one-dimensional, the local divergence rate is given by

$$\lambda_{i,T} = \frac{1}{T}\frac{1}{N_i}\sum_{j}^{N_i} log|\frac{x_{i+T} - x_{j+T}}{x_i - x_j}| \tag{1}$$

where $\lambda_{i,T}$ is the Lyapunov exponent associated with the point x_i and the lag T, N_i being the number of points within a distance h of x_i, where h is the radius of a hyper-sphere centered in x_i.

In the case of multi-dimensional systems, the attractor is reconstructed using the method of delays [6], and to each point x_n is associated a vector in phase space X_n

$$X_n = [x_n \ x_{n+\tau} \ \cdots \ x_{n+(m-1)\tau}] \tag{2}$$

where m is the embedding dimension, and τ the embedding delays that are necessary in order to correctly reconstruct the attractor.

The process of estimating the local divergence rate then consists in computing the logarithm of the norm of the difference instead of the logarithm of the absolute value of the difference as in eq.(1) :

$$\lambda_{i,T} = \frac{1}{T}\frac{1}{N_i}\sum_{j}^{N_i} log\frac{||X_{i+T} - X_{j+T}||}{||X_i - X_j||} \tag{3}$$

N_i then becomes the number of points within a hyper-sphere of radius h centered in X_i. An example of results is shown on figure 1.

It is clear from this figure that it is difficult to extract meaningful information from such a computation. It is necessary to group the exponents into regions in order to be able to analyze them.

2.2 Clustering of the Exponents

Clustering techniques have been used for a long time in the image processing and pattern recognition areas, and elaborate methods have been developed (cf [11] for a recent review on the subject). Clustering is a rather fuzzy term, which groups all methods assembling in the best possible way elements according to given characteristics.
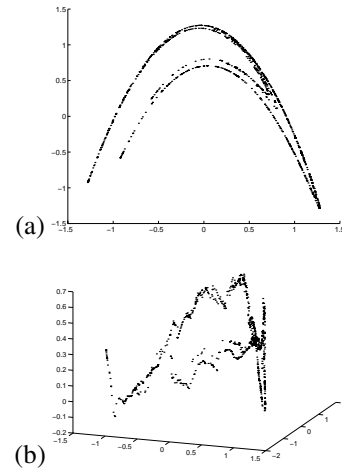

FIG. 1 — Example of local exponents results on the Henon attractor. a) Henon attractor itself. b) Values of the local exponents *vs* their localization in phase space

Many clustering methods exist, having either a top-bottom (divide into subgroups) or bottom-up approaches (grouping of elements), using supervised or unsupervised schemes, hierarchical or non-hierarchical approaches. Here, the method that is looked for is a method that needs no human supervision, that builds meaningful clusters from the individual points in phase space, clusters having a physical/logical meaning. The approach should thus be unsupervised, bottom-up, and non-hierarchical.

There are methods building clusters by agglomerating points without any supervision or *a priori* knowledge on the number of clusters, whose most advanced versions are the c-means and fuzzy clusterings ([12, 13] and refs therein). They are however not of much use here, since all require a limited number of characteristics, which themselves can take only a limited number of values. The clustering is thus made as follows :

The characteristics of choice in this work are a) the local Lyapunov exponents, and b) the position in space. The use of the latter is necessary in order to avoid creating clusters that group points having no relationship in space, such as clusters at both extremes of the attractor that have the same exponent but do not share a common border. The clustering on these characteristics is then performed :

1) Split the range of values for $\lambda_{i,T}$ into K intervals of equal size, K being set by the user. This defines K different exponents of value $\bar{\lambda}_k$.

2) For each point on the attractor, define a hyper-sphere of radius r which will be the tentative cluster.

3) Within this hyper-sphere, look for all points having the same exponent $\bar{\lambda}_k$ as the reference point. If they are not separated by points having a different exponent, they are considered to be part of the same cluster.

4) If a point in the hyper-sphere is found to have the same $\bar{\lambda}_k$ than the reference point, but is already part of a cluster, the latter and the cluster of the reference point are merged toge-

ther.

While this procedure is unsupervised and does not require the setting of a final number of clusters, this number depends on parameters set by the user. The first one is the size of the hyper-sphere h that is chosen for the exponent estimation : It rules the smoothness of the local Lyapunov exponent by increasing or lowering the number of neighbors that are taken into account for the computation.

The other critical parameter is the number K of intervals that are chosen for the "digitalization" of the exponents. It does not set the final number of clusters, but gives an order of magnitude. There can indeed be clouds of points which have the same exponent $\bar{\lambda}_k$, but which are not connected together. Conversely, it may happen that there is no exponent of a given value, which will automatically decrease the number of clusters.

3   Results

The simulation results show that this technique is relevant in that it shows the actual local dynamics of the attractor. This is well illustrated by the example of the Lorenz attractor on fig. 2
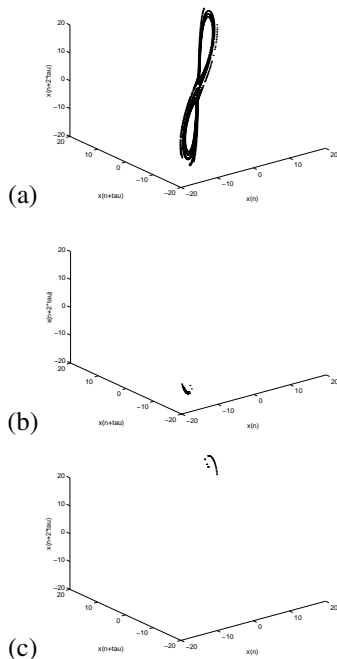

(a)

(b)

(c)

FIG. 2 — Main clusters of the Lorenz attractor for $h = 0.7$, $t = 0.15$, $N = 2000$ points, 9 clusters. a) Main cluster, b) second biggest cluster, c) third biggest cluster

It is interesting to note that the 3 main clusters already span most of the attractor, and that this remains true for a wide range of parameters. The separations between clusters seem intuitively logical : the center part of the attractor is characterized by quickly diverging trajectories, while the end of the 'wings' of the attractor are characterized by more parallel curves. If the number of clusters is increased, the center part of the attractor is rapidly split into two separate wings, with a central part in-between. The next figure shows the three main clusters of the same invariant set.
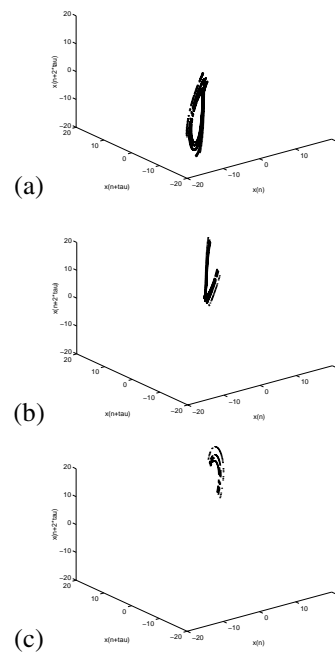

(a)

(b)

(c)

FIG. 3 — Main clusters of the Lorenz attractor for $h = 0.5$, $t = 0.15$, $N = 2000$ points, 20 clusters

Simulations were run for other chaotic signals, such as the Henon and Ikeda maps, but the results are not displayed here, since they are essentially similar to those of the Lorenz system.

4   Discussion

The results show that this natural clustering of the exponents is useful : The results are intuitively logical, but its results can also be validated from another point of view.

This clustering technique can indeed be compared to the so-called symbolic dynamics [14, 15], in the sense that it builds a kind of table of symbols for given regions on the attractor. However, it is different in the sense that it is applied on a measure of the attractor instead of the actual area of the attractor. Moreover, the split up of the attractor is not governed by an arbitrary decision, but by laws of the system itself. It can be considered as a variation on the symbolics : symbols are still given to a regions of the invariant set, but the number of symbols (i.e. the number of clusters) is decided by the data themselves instead of the user.

It is also very interesting to compare the results we obtain with the ones of Mattavelli *et al* in [16]. In their article, they present a method for selecting piecewise linear models on a nonlinear system. The main idea of their paper is to focus on the inconsistent linear system that arises when considering a simple linear model and to partition it into a minimum number of subsystems. The results they present are very consistent with the results seen on fig. 2 and 3. Most of the differences between their partition and ours arise from the fact that for

us two clusters having the same mean Lyapunov exponents separated by another cluster are kept separate, while they consider the regions corresponding to identical models in their graphs.

In its present state, the algorithm suffers from several drawbacks, one of the most important being that only qualitative analysis of the Lyapunov exponents is possible. Due to the imprecision of the computation, the numerical value of the exponent is not reliable, and the global largest exponent computed from the local ones does not converge to the expected value, at least for reduced sets of data. However, this does not prevent a qualitative analysis and a correct clustering, since the local variations are preserved. Another drawback of the algorithm is its computational load due to the relatively inefficient clustering technique. A next step should be to search for more efficient algorithms both for the local exponent computation and clustering parts.

It is to be noted that while the clustering technique has been applied on local Lyapunov exponent, it is by no means limited to this sole value. It could be applied with profit on a variety of other measures, such as the local intrinsic dimension, or local spatial dimension, or as a 'natural' way of computing the symbolic sequences associated with a signal. The informations obtained form these measures should give complementary informations, thus leading to a better understanding of the local behavior of a signal, and possibly helping in selecting piece-wise models.

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