4D Tomography: an Application of Incremental Constraint Projection Methods for Variational Inequalities

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Résumé – La tomographie 3D + temps, ou 4D, peut être vue comme un problème d'optimisation convexe. Tout revient à rechercher le minimum d'une fonction de coût, qui peut inclure de l'information connue *a priori*, comme des contraintes de support ou de parcimonie dans une base donnée. La taille du jeu de projections X et de l'image 4D à reconstruire, ainsi que les contraintes de temps de calcul pour permettre une utilisation clinique, restreignent fortement le choix de l'algorithme de minimisation : la matrice du système est trop grande pour être stockée explicitement, il faut éviter autant que possible les copies multiples de l'image 4D, et l'algorithme doit converger (ou presque) en quelques itérations. Cette dernière contrainte suggère l'utilisation d'un algorithme incrémental, comme la méthode de Kaczmarz ou Kaczmarz par blocs, car les algorithmes incrémentaux ont souvent une convergence initiale plus rapide que leurs homologues non-incrémentaux. Ce travail décrit comment une méthode existante qui satisfait toutes ces conditions peut être appliquée à la tomographie 4D régularisée. Il évalue l'impact de l'utilisation d'une approche incrémentale sur la vitesse de convergence en fonction du nombre de sous-ensembles choisi. Les résultats préliminaires sur un fantôme 2D + temps sont encourageants, et une implémentation 3D + temps adaptée au traitement de données réelles est en cours.

Abstract – 3D + time, or 4D, X-ray tomography can be cast into a convex optimization problem. It amounts to seeking the minimum of a carefully chosen cost function, which can include *a priori* information like support constraints or sparsity on a relevant basis. The size of both the X-ray projections and the sought 4D image, as well as the maximum computation time that can be tolerated in a clinical environment, severely restrict the choice of the minimization algorithm: the system matrix is too large to be stored, multiple copies of the sought 4D image must be avoided, and convergence should require few iterations. The latter requirement suggests to use an incremental method, for example the Kaczmarz or block-Kaczmarz method, as incremental methods often have faster initial convergence than their non-incremental counterparts. This paper describes how an existing method meeting these requirements can be applied to regularized 4D tomography. It evaluates the impact of using an incremental approach on the convergence speed, depending on the chosen number of subsets. Preliminary results on 2D + time phantom data are encouraging, and the 3D + time implementation for real data will follow.

1 Introduction

4D X-ray cone beam computed tomography is an active topic of research in both cardiac and pulmonary imaging. A robust and fast method for this purpose would find various applications, including in-room planning and guidance for interventional cardiology procedures and targeted radiotherapy of lung tumors. 4D X-ray tomography can be formulated as the minimization of a sum of convex functions, some of which are not differentiable. Proximal splitting methods seem well-suited for this kind of problem. Yet, this type of methods solve the problem in a product space [3, 4] and require to store multiple copies of the sought 4D image. Computation time is a major issue in a clinical context. As the performance bottleneck is usually in the forward or back projection operator, methods that require fewer uses of these operators tend to be faster. A classical way to achieve good convergence within a limited number of iterations is the Kaczmarz or block-Kaczmarz method, which in tomography are applied as the Algebraic Reconstruction Technique (ART) [5] and the Simultaneous Algebraic Reconstruction Technique (SART) [1]. These "incremental" methods achieve fast initial convergence [2], which is a desirable property when the number of iterations is bound to remain small due to computation time constraints. This paper describes how an incremental constraint projection method for variational inequalities proposed by Wang and Bertsekas [8] can be applied to adapt a Total Variation regularized 4D cone beam computed tomography (4D CBCT) method, namely the 4D RecOnstructiOn using Spatial and TEmporal Regularization (4D ROOSTER) [6]. It compares the convergence rates obtained by the incremental method with various numbers of projections subsets.

2 Material and methods

2.1 The original 4D ROOSTER method

The 4D ROOSTER algorithm assumes that a rough segmentation of the patient's rib cage is available, and that movement is expected to occur only inside this segmented region. The method consists in alternating between five different optimization goals. It starts by minimizing a quadratic data-attachment term $\sum_{\theta} \|R_\theta S_\theta x - p_\theta\|_2^2$, with θ the projection angle, x a 4D sequence of volumes, R_θ the forward projection operator at angle θ, S_θ a linear interpolator, and p_θ the measured projection at angle θ. This data-attachment term is minimized by conjugate gradient. Then the following regularization steps or constraints are applied sequentially: positivity enforcement, averaging along time where motion is not expected, spatial total-variation (TV) denoising, and temporal total-variation denoising. This constitutes one iteration of the main loop, the output of which is fed back to the conjugate gradient minimizer for the next iteration. Although this method yields good results in practice, no formal proof of its convergence is available.

2.2 Incremental 4D ROOSTER

The adaptation of 4D ROOSTER we present here aims to solve the following optimization problem:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \sum_{\theta \in \Theta} \|R_\theta S_\theta x - p_\theta\|_2^2 \\ & + \lambda_{space} TV_{space}(x) \\ & + \lambda_{time} TV_{time}(x) \\ \text{subject to } & x \geq 0 \\ & x \in ROI \end{aligned}$$

where ROI is the convex set of all 4D images such that only voxels in the segmented region vary over time. In order to make the method incremental, the data-attachment term is first divided into n random subsets of projections, as balanced as possible, which we name $\Theta_i,\ i = 1, \ldots, n$, such that $\cup_{i=1}^n \Theta_i = \Theta$ and $\Theta_i \cap \Theta_j = \emptyset$ for $i \neq j$. Setting

$$\begin{aligned} f_i(x) &= \sum_{\theta \in \Theta_i} \|R_\theta S_\theta x - p_\theta\|_2^2, \text{ for } i = 1, \ldots, n \\ f_{n+1}(x) &= \lambda_{space} TV_{space}(x) \\ f_{n+2}(x) &= \lambda_{time} TV_{time}(x) \end{aligned}$$

the problem becomes

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \sum_{i=1}^{n+2} f_i \\ \text{subject to } & x \geq 0 \\ & x \in ROI \end{aligned}$$

One instance of the class of algorithms described in Wang & Bertsekas [8] to solve the incremental version of the problem is the following:

for $k = 1 \ldots$ NbSteps

$\alpha_k = \frac{k_0}{k + 2k_0}$
Select one of the f_i and compute $\hat{x} = \text{prox}_{\alpha_k f_i}(x_k)$
Select one of the constraints and project onto it: $x_{k+1} = \prod_{C_j}(\hat{x})$

end

The f_i and the constraint are selected by deterministic cycling, which is one of the ways to ensure the convergence of the algorithm. This allows to distinguish between "steps" and "iterations": in the rest of this paper, an "iteration" of the algorithm means $n + 2$ "steps", i.e. it implies the processing of all projection subsets (n steps), plus the spatial and temporal TV denoising steps. Instead of computing the $\text{prox}_{\alpha_k f_i}(x_k)$ for the data-attachment terms, we chose to compute $\hat{x}$ by performing a few nested iterations of conjugate gradient (CG), which corresponds to a regularized solution that is not far from the proximal solution. We start with 4 nested iterations, which proved efficient in previous studies [6].

In section 3.1, we compare the convergence speed, in terms of Root Mean Square Error (RMSE) with the ground truth, of several instances of the incremental 4D ROOSTER method, with different values of the number of subsets n. The number of iterations NbIter is first kept constant between instances. A threshold on the RMSE is then used as a stopping criterion, which for some instances reduces the number of iterations actually performed .

In section 3.2, we try modifying the number of nested CG iterations instead of modifying NbIter.

2.3 Phantom data

Preliminary results are presented on a 2D + time phantom, adapted from a high-contrast Shepp & Logan phantom. One of the ellipses shrinks and dilates over time to roughly mimic a beating heart, and the acquisition conditions simulate a 4D cardiac CBCT of a patient, with the following parameters:

- 300 parallel projections over 180°
- Heart rate: 72 beats per minute
- Total acquisition time: 10 seconds (30 projections per second)
- x is a 128×128 pixels $\times 10$ frames image

3 Results

3.1 Adapting the number of iterations

We first compare the convergence speed, in terms of RMSE, of several instances of the incremental 4D ROOSTER method, with different values of the number of subsets n. Figure 1 shows the RMSE as a function of the number of iterations. It clearly shows an increase of convergence speed when the number of subsets increases. From figure 1, we define a threshold (here, 100) and re-run all versions of the incremental ROOSTER, stopping the iterations when the RMSE with the ground truth gets under the threshold. Table 1 shows the number of iterations performed in each case. Figure 2 shows a single frame of the 2D + time reconstruction each instance of the incremental ROOSTER yields, and the ground truth. It allows to visually evaluate the quality of the reconstructions. Figure 2 shows that, with comparable RMSE, the reconstruction results obtained with higher values of n are noisier around the moving ellipse (we remind that the rest of the voxels are averaged along time, as a result of one of the constraints, which explains their low noise). Two possible causes can be identified to explain this higher noise:

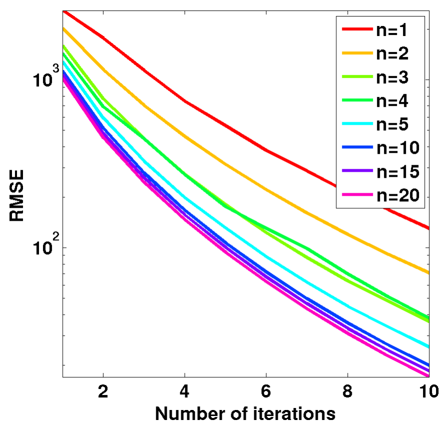

Figure 1: RMSE with the ground truth of the phantom, as a function of the number of iterations, depending on the number of subsets n

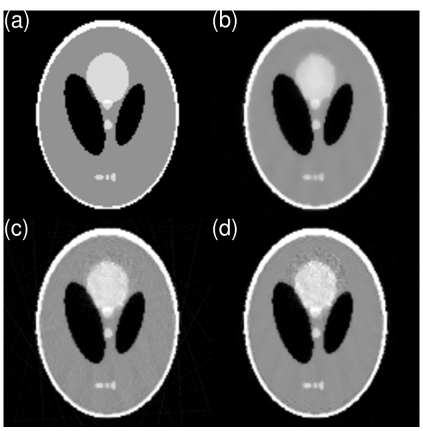

Figure 2: Incremental ROOSTER reconstructions with RMSE threshold = 100. (a) Ground truth (b) $n = 1$ (c) $n = 5$ (d) $n = 20$

Table 1: Number of iterations performed by each instance of incremental with an RMSE threshold of 100

Subsets	1	2	3	4	5	10	15	20
Iterations	10	9	7	7	6	6	5	5

- incremental methods typically favor high frequencies, which may amplify the noise
- Total Variation regularization was performed 10 times with $n = 1$, and only 5 times with $n = 20$

While little can be done to mitigate the former effect, the latter can be avoided by using a different strategy to take advantage of the faster convergence of incremental methods: instead of reducing the main number of iterations, we can adapt the number of nested CG iterations performed at each step.

3.2 Adapting the number of nested CG iterations

We now compare the convergence speed and reconstruction results of the $n = 1$ instance with 4 nested CG iterations and the $n = 20$ instance with 1 to 4 nested CG iterations. Figure 3 shows the compared convergence speeds of non-incremental ROOSTER ($n = 1$, 4 nested CG iterations) and incremental ROOSTER with $n = 20$ and 1 to 4 nested CG iterations. Figure 4 shows the reconstruction results. Even with a single nested CG iteration,

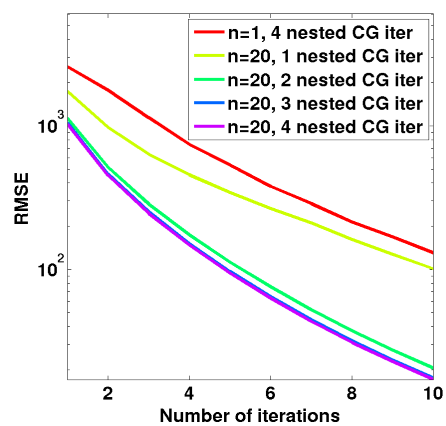

Figure 3: RMSE with the ground truth of the phantom, as a function of the number of iterations, depending on the number of subsets n

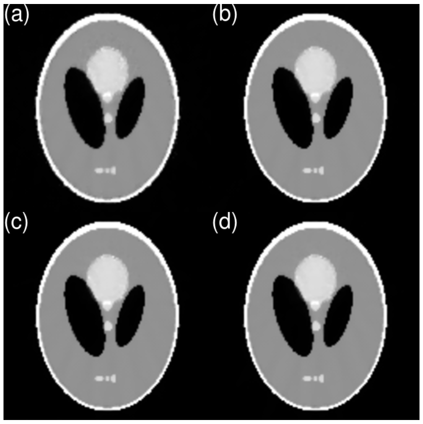

Figure 4: Incremental ROOSTER reconstructions with 10 iterations, $n = 20$ and a variable number c of nested CG iterations. (a) $c = 1$ (b) $c = 2$ (c) $c = 3$ (d) $d = 4$

the $n = 20$ instance achieves faster convergence than non-incremental ROOSTER. Visual evaluation confirms the gain in image quality with respect to the first approach.

4 Discussion

The λ_{space} and λ_{time} parameters, as well as the formula for computing α_k, have been chosen empirically. λ_{space} and λ_{time} depend on the expected visual aspect of the solution, and on the size and resolution of the reconstructed image. Previous work on ROOSTER [6] shows that for a given size and resolution, a pair of λ_{space} and λ_{time} that yields clinically satisfying results on one patient also works well on another patient. The sequence of α_k must be such that $\sum_{k=0}^{\infty} \alpha_k = +\infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < +\infty$.

The speedup obtained by reducing the total number of forward and back projections for the same convergence must not be overestimated: conjugate gradient iterations involve operations on the 4D image (sum of all voxels and voxelwise operations), which in the incremental approach are performed n times, adding some computation time.

5 Conclusion

Incremental 4D ROOSTER builds up on solid mathematical foundations, has a convergence proof of the function values and has the potential to yield results visually similar to those of the original 4D ROOSTER within fewer forward and back projections. The preliminary results, obtained with Matlab on 2D + time data, are promising. Incremental 4D ROOSTER will be implemented in “The Reconstruction ToolKit” (RTK) [7], an open source C++ software based on “The Insight ToolKit” (ITK), and should be compared with a proximal strategy in some future work.

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