Achievable Rates of Additive Symmetric α-Stable Noise Channels

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Résumé – Un bruit impulsif est observé dans de nombreux systèmes de communication modernes. Il peut souvent être modélisé par une distribution α-stable. Actuellement, la capacité des canaux à bruit additif α-stable n'est pas bien comprise, à l'exception du cas $\alpha = 1$ avec une contrainte logarithmique. Dans cet article, nous considérons le canal à bruit additif α-stable symétrique avec $\alpha \in (1, 2]$, soumis à une contrainte du moment de la valeur absolue. Nous présentons un nouveau débit atteignable et proche de la capacité. Nous étudions son comportement pour des valeurs modérées de la contrainte (analogue au rapport signal sur bruit), ainsi que la probabilité de coupure pour un canal bande étroite de Rayleigh.

Abstract – Impulsive noise arises in many modern communication systems and is often modeled by the α-stable distribution. At present, the capacity of α-stable noise channels is not well understood, with the exception of $\alpha = 1$ with a logarithmic constraint. In this paper, we consider additive symmetric α-stable noise channels with $\alpha \in (1, 2]$ subject to an absolute moment constraint. We present a new and tight achievable rate and investigate its behavior for moderate values of the constraint (analogous to medium signal-to-noise ratio), as well as the outage probability with Rayleigh fading.

1 Introduction

Impulsive noise arises in underwater [1], wireless [2], power line [3] and molecular [4] communications. Characterized by a higher probability of large amplitude noise, impulsiveness is not well captured by Gaussian models.

General models of impulsive noise are due to Middleton [5], which were derived for interference in wireless networks from a statistical physics perspective. As it has proven difficult to characterize the capacity of channels with Middleton's noise models, approximate models have been introduced. A key class of these models are the symmetric α-stable distributions, which can be viewed as generalizations of zero-mean Gaussian models ($\alpha = 2$) and preserving stability for independent random variables. This approach leads to the additive symmetric α-stable noise ($AS\alpha SN$) channel, where the scalar output Y is given by

$$Y = X + N, \tag{1}$$

with input signal X and symmetric α-stable noise N.

A challenge for the design of communication systems in the presence of impulsive noise is that the capacity of the $AS\alpha SN$ channel is not well understood. The difficulty in characterizing the capacity of $AS\alpha SN$ channels is in part due to the fact that a power constraint $\mathbb{E}[X^2] \leq P$ is typically imposed and, unlike the Gaussian case, the second moment of α-stable distributions is infinite for $\alpha < 2$. At present, Fahs and Abou-Faycal [6] have studied the structure of the optimal input distribution for logarithmic and fractional moment constraints. However, there are currently no characterizations of the capacity for $AS\alpha SN$ channels with $0 < \alpha < 2$.

In this paper, we study the capacity of the $AS\alpha SN$ channel subject to an absolute moment constraint. In particular, we consider the constraint

$$\mathbb{E}[|X|] \leq c, \quad c > 0. \tag{2}$$

A key feature of this constraint is that it admits tractable achievable rates for $\alpha \in (1, 2]$. More precisely, the capacity of the $AS\alpha SN$ channel is lower bounded by

$$C \geq \frac{1}{\alpha} \log_2 \left(1 + \left(\frac{c}{\mathbb{E}[|N|]}\right)^{\alpha}\right), \tag{3}$$

obtained by matching the input, X, and noise, N as α-stable distributions. Numerical comparisons with the capacity approximations from the Blahut-Arimoto algorithm show that the bound is tight, particularly for $\alpha \approx 2$.

To understand how the achievable rates in (3) behave for different values of α—particularly for the Gaussian case ($\alpha =$

2)—we investigate its behavior for moderate values of the constraint (analogous to the medium signal-to-noise ratio regime). This is achieved by computing the bend point, introduced in [7]. The bend point provides insight into the beginning of the high c region, analogous to the signal-to-noise ratio (SNR) for the Gaussian channel. We also derive an upper bound on the outage probability in the presence of Rayleigh fading and the input distribution parametrization. This analysis suggests that our capacity lower bound may be useful as a performance metric for systems with symmetric α-stable noise.

The remainder of the paper is organized as follows. In Section 2, we summarize the α-stable random variables and we show the achievable rates. In Section 3, we study further applications of our lower bound. In particular, we derive medium c properties and an outage probability bound in the presence of fading. In Section 4, we conclude and provide avenues for future work.

2 Achievable Rate Analysis

In the section, we present new achievable rates for the $AS\alpha SN$ channel defined in (1). In order to obtain our result, we first recall the notion of symmetric α-stable random variables which forms the basis of the noise, N. The achievable rates are then obtained as lower bounds for the capacity of the $AS\alpha SN$ channel, for which a precise definition can be found in [8, Section II]. We show that the achievable rates form tight lower bounds of the capacity via a numerical comparison with the approximate capacity obtained using the Blahut-Arimoto algorithm.

2.1 The α-Stable Random Variables

The α-stable random variables are heavy-tailed probability density functions [9]. The probability density function of an α-stable random variable is parameterized by four parameters: the exponent $0 < \alpha \leq 2$; the scale parameter $\gamma \in \mathbb{R}_+$; the skew parameter $\beta \in [-1, 1]$; and the shift parameter $\delta \in \mathbb{R}$. As such, a common notation for a general α-stable distributed random variable is $N' \sim S_\alpha(\gamma, \beta, \delta)$. In the case $\beta = \delta = 0$, the random variable N is a symmetric α-stable random variable denoted by $N \sim S_\alpha(\gamma, 0, 0)$.

However, the additive symmetric α-stable noise, N' usually does not have a closed form. Instead, they are represented by their characteristic function, given by

$$\mathbb{E}[e^{i\theta N'}] = \begin{cases} \exp\left\{-\gamma^\alpha|\theta|^\alpha(1 - i\beta(\text{sign}\theta)\tan\frac{\pi\alpha}{2}) + i\delta\theta\right\}, & \alpha \neq 1 \\ \exp\left\{-\gamma|\theta|(1 + i\beta\frac{2}{\pi}(\text{sign}\theta)\log|\theta|) + i\delta\theta\right\}, & \alpha = 1 \end{cases}. \tag{4}$$

In turn, the channel considered in this paper given by (1) is restricted to the additive symmetric α-stable distributed random variable N denoted by

$$\mathbb{E}[e^{i\theta N}] = \exp\left(-\gamma^\alpha|\theta|^\alpha\right), \ \theta \in \mathbb{R}. \tag{5}$$

2.2 Achievable Rates

Achievable rates of the $AS\alpha SN$ channel are given by the following theorem presented in (3). The result is obtained by using a codebook consisting of codewords distributed according to the symmetric α-stable distribution; that is, the input X is matched to the noise N. For full details, see [8].

Theorem 1. *Achievable rates of the $AS\alpha SN$ channel subject to an absolute moment constraint $\mathbb{E}[|X|] \leq c$ with $N \sim S_\alpha(\gamma, 0, 0)$ and $1 < \alpha < 2$ is given by [8]*

$$\begin{aligned} C &\geq \frac{1}{\alpha}\log_2\left(1 + \frac{\gamma_X^\alpha}{\gamma_N^\alpha}\right) \\ &\geq \frac{1}{\alpha}\log_2\left(1 + M_\alpha\left(\frac{c}{\gamma_N}\right)^\alpha\right), \end{aligned} \tag{6}$$

where

$$M_\alpha = \left(\frac{\pi}{2\Gamma\left(1 - \frac{1}{\alpha}\right)}\right)^\alpha, \tag{7}$$

using the constraint in (2).

To validate the tightness of the achievable rates, Fig. 1 compares our bound and the numerical capacity approximation using the Blahut-Arimoto algorithm for $\alpha = 1.9$. This provides strong evidence that the achievable rates are in fact tight for α near 2, although this behavior is not observed when small values are used, with more distant rates as long as α decreases.

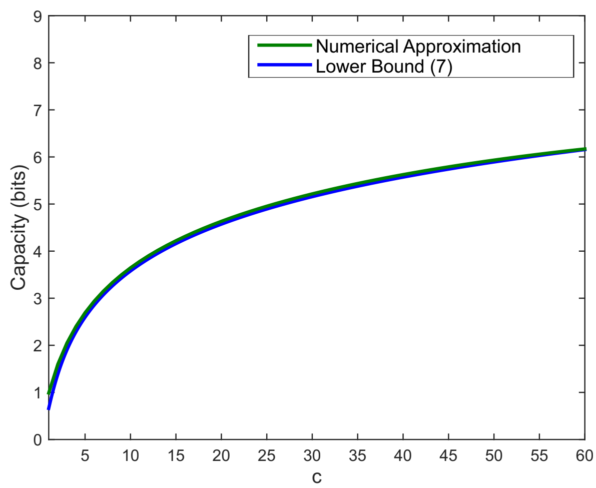

Figure 1: Plot of our capacity lower bound for varying c, with $\alpha = 1.9$, $\gamma_N = 1$, $\beta = 0$ and $\delta_N = 0$

3 Behavior of the Achievable Rates

We now study the behavior of the achievable rates for moderate values of the constraint, c, and in the presence of fading. The tractability of the achievable rate means that it is an attractive performance metric in settings based on the $AS\alpha SN$ channel, and can play a role similar to the power constrained capacity in settings based on the Gaussian channel.

3.1 Medium c Behavior

For moderate values of the constraint, c, a useful notion is the bend point. Introduced in [7] in the context of Gaussian noise channels, the bend point quantifies the transition of the achievable rates. The bend point corresponds to the point on the achievable rate curve where the rate of change of the slope is maximized (analogous to the medium signal-to-noise (SNR) regime). For the $AS\alpha SN$ channel, we define the bend point as follows.

Definition 1 (Bend point)**.** *Consider the achievable rates in Theorem 1, given by*

$$C_{LB} = \frac{1}{\alpha}\log_2\left(1 + M_\alpha\left(\frac{c}{\gamma_N}\right)^\alpha\right), \tag{8}$$

where M_α is given by (7).

The bend point, c_{bend}, is then the $c_{dB} = 10\log_{10} c$ such that the second derivative of (8) is maximized.

As such, it can be viewed as the transition between high and low c as the rate of change of the slope tends to zero as $c_{dB} \to -\infty$, reaches its maximum value at the bend point, and then tends to zero as $c_{dB} \to \infty$.

A key observation in [7] is that the bend point is intimately related to the intersection of high and low SNR asymptotes in the capacity of power constrained Gaussian channels. We now investigate the bend point in the context of the $AS\alpha SN$ channel.

Theorem 2. *The bend point is given by*

$$c_{bend} = \frac{10}{\alpha}\log_{10}\left(\frac{\gamma_N^\alpha}{M_\alpha}\right). \tag{9}$$

Proof. The third derivative of (8) in Theorem 1 is given by

$$C'''_{LB} = \frac{M_\alpha\left(\frac{\alpha}{10}\log 10\right)^3}{\alpha\gamma_N^\alpha \log 2}\left[\frac{10^{\alpha c_{db}/10}\left(1 - \frac{M_\alpha}{\gamma_N^\alpha}10^{\alpha c_{dB}/10}\right)}{\left(1 + \frac{M_\alpha}{\gamma_N^\alpha}10^{\alpha c_{dB}/10}\right)^3}\right], \tag{10}$$

which satisfies $C'''_{LB} = 0$ when $c_{dB} = \frac{10}{\alpha}\log_{10}\left(\frac{\gamma_N^\alpha}{M_\alpha}\right)$. Note also that the second derivative of (6) is given by

$$C''_{LB} = \frac{M_\alpha\left(\frac{\alpha}{10}\log 10\right)^2}{\alpha\gamma_N^\alpha \log 2}\left[\frac{10^{\alpha c_{db}/10}}{\left(1 + \frac{M_\alpha}{\gamma_N^\alpha}10^{\alpha c_{dB}/10}\right)^2}\right], \tag{11}$$

and is symmetric around $c_{dB} = 0$ and decreasing for $c_{dB} > 0$, which proves the theorem. □

Now, define the asymptote (as $c_{dB} \to \infty$) of the lower bound as

$$C_{asymp} = \frac{1}{\alpha}\log_2\left(10^{\alpha c_{dB}/10}\right) + \frac{1}{\alpha}\log_2\left(\frac{M_\alpha}{\gamma_N^\alpha}\right). \tag{12}$$

Note that this asymptote describes the behavior of the achievable rate for large values of c_{dB} (equivalently, c). Observe that although the asymptote varies as α changes, the achievable rate curve is linear when plotted against c_{dB} which is consistent with the capacity of the power constrained Gaussian noise channel.

A further observation is that $C_{asymp} = 0$ when $c_{dB} = \frac{10}{\alpha}\log_{10}\left(\frac{\gamma_N^\alpha}{M_\alpha}\right)$, which agrees with the bend point c_{bend}, from Theorem 2. This means that as for the power constrained Gaussian channel, the intercept asymptote of the achievable rate curve for the $AS\alpha SN$ channel agrees with the bend point; however, unlike the power constrained Gaussian channel, the bend point does not always occur at $c_{dB} = 0$.

Fig. 2 plots the capacity lower bound for varying α. Observe that the bend point c_{bend} is reduced as α increases. This suggests that using the asymptotic approximation is more accurate for Gaussian channels than for the $AS\alpha SN$ channel at lower values of c. As asymptotic approximations are widely used, this implies that approximations that are valid in the Gaussian case are likely to be less accurate for other values of α.

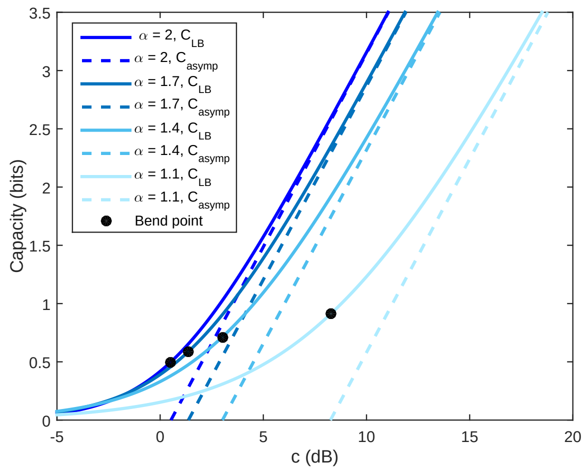

Figure 2: Plot of our capacity lower bound, C_{LB} for varying α, with $\gamma_N = 1$, $\beta = 0$ and $\delta_N = 0$. The dot on each curve is the corresponding bend point.

3.2 The Effect of Fading

Fading plays an important role in many Gaussian channels. In the case of the additive α-stable noise, we obtain the channel model given by

$$Y = gX + N, \tag{13}$$

where g represents the real-valued fading coefficient. For a fixed g, the capacity constraint in (2) is lower bounded by

$$C \geq \frac{1}{\alpha}\log_2\left(1 + |g|^\alpha M_\alpha \frac{c^\alpha}{\gamma_N^\alpha}\right), \tag{14}$$

which follows from Theorem 1, where M_α is defined in (7).

In the case of slow fading (g varies slowly, but randomly according to a fixed distribution F_g), the transmission quality is often characterized by the outage probability, which is given by

$$\begin{aligned} P_{out} &= \Pr(C \leq R_0) \\ &\leq \Pr\left(\frac{1}{\alpha}\log_2\left(1 + |g|^\alpha M_\alpha \frac{c^\alpha}{\gamma_N^\alpha}\right) \leq R_0\right). \end{aligned} \tag{15}$$

A common choice for the distribution of g^2 is $F_{g^2}(x) = 1 - e^{-\lambda x}$, corresponding to Rayleigh fading. The outage probability is then bounded by

$$\begin{aligned} P_{out} &\leq \Pr\left(|g| \leq \frac{\gamma_N}{M_\alpha^{\frac{1}{\alpha}} c}\left(2^{\alpha R_0} - 1\right)^{\frac{1}{\alpha}}\right) \\ &= 1 - \exp\left[-\lambda\left(\frac{\gamma_N^2}{M_\alpha^{\frac{2}{\alpha}} c^2}\left(2^{\alpha R_0} - 1\right)^{\frac{2}{\alpha}}\right)\right] \\ &= P_{LB} \end{aligned} \tag{16}$$

Fig. 3 demonstrates the effect of R_0 on the outage upper bound. The simulated curves are based on Monte-Carlo simulations over 1000 realizations of the fading channel. The exact curve is obtained from (16). Observe that for small R_0 the outage probability upper bound is heavily influenced by α.

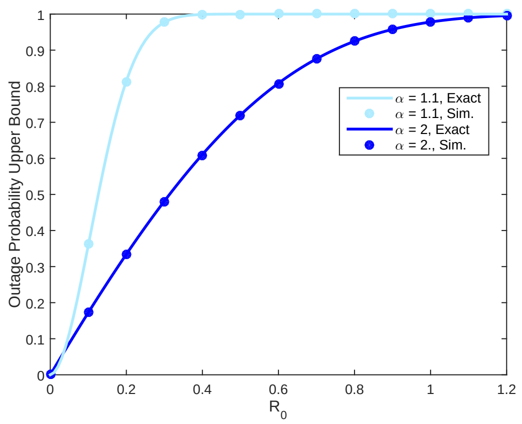

Figure 3: Plot of the outage probability upper bound (16) for varying R_0 and α, with $\beta = 0$, $c = \gamma_N = 1$, $\delta_N = 0$, and $\lambda = 1$.

4 Conclusion

Impulsive noise plays a key role in many communication systems, ranging from wireless to molecular. Important models for impulsive noise are the symmetric α-stable distributions. In this paper, we use a tractable lower bound for the $AS\alpha SN$ channel, with $\alpha \in (1, 2]$. We have investigated its behavior in the medium c regime and the effect of fading.

There are several avenues for future work. In particular, the case of $0 < \alpha \leq 1$, and asymmetric α-stable noise distributions remain open. The tractability of our lower bound and its close relationship to the capacity of Gaussian noise channel with a power constraint also suggests that it may be able to play an analogous role in more applications. For instance, this opens the question of the behavior and design of algorithms for parallel and MIMO additive α-stable noise channels.

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