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A COMPARISON BETWEEN WIGNER-VILLE AND EVOLUTIONARY SPECTRA FOR COVARIANCE EQUIVALENT NONSTATIONARY RANDOM PROCESSES

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RESUME

Des processus aleatoires non stationnaires, dont la configuration est modulee en frequence, se pregentent dans la pratique: on peut citer l'exemple de la perception, par un observateur, de son emis par une source en mouvement produisant un signal aleatoire. Le signal considere dans le repere de reference de la source peut paraitre stationnaire, alors que, considere dans un repere lie a l'observateur (en mouvement relatif par rapport a la source) il n'apparait plus stationnaire du fait de la 'dilatation' de la variable independante.

Cet article decrit la caracterisation en temps-frequence de tels processus en termes de spectre de Wigner-Ville et de spectre evolutionnaire de Priestley.

Pour calculer les densites spectrales evolutionnaires des processus modules en frequence, le concept de 'equivalence de covariance' est necessaire.

On montre que les deux caracteristiques spectrales sont liees et on utilise, pour demonstrer la relation entre ces dernieres quantites, l'exemple d'une source monopolaire se deplacant a vitesse constante.

SUMMARY

Nonstationary random processes having a frequency modulated form arise in practice, for example, the sound perceived by an observer due to a source that is moving and emitting a random signal. The signal, when viewed from the frame of reference of the source may be stationary but when viewed by the observer in motion relative to the source it appears nonstationary due to the 'dilation' of the independent variable. This paper describes the time-frequency characterisation of such processes in terms of the Wigner-Ville and Priestley's evolutionary spectra. In order to calculate evolutionary spectral densities for frequency modulated processes the concept of 'covariance equivalence' is necessary.

The two spectral characterisations are shown to be related and an example of a simple monopole source travelling at constant speed is used to demonstrate the relationship.

INTRODUCTION

Some physical nonstationary random processes exhibit a 'frequency modulated' structure. An example of such a process is the sound perceived by an observer due to a moving sound source emitting a random signal. If such random processes are viewed in the 'frame of reference' of the source they may be stationary and the acoustic signal emitted by the moving source is stationary to an observer moving with the source. However, when such processes are viewed by an observer in motion relative to the source, then a dilation of the independent variable characterising the process occurs and the process is nonstationary in the frame of reference of the observer and is frequency modulated in form.

The description of spectra for such processes is of interest and two important candidates for 'time-frequency' spectral characterisation of nonstationary processes are the Wigner-Ville spectrum [1,2] and Priestley's evolutionary spectral density [3].

A technique called the 'covariance equivalent' method has been put forward [4,5,6] to describe such processes. In [7] the authors have shown how evolutionary spectra and Wigner-Ville spectra may be related for covariance equivalent processes. In this paper the work reported in [7] is summarised and the linking of the two forms of time-frequency spectra made more explicit. These results provide a basis for the interpretation of the results of an example, namely the description of the sound perceived by an observer due to a moving monopole source which emits a random signal.

WIGNER-VILLE SPECTRA AND EVOLUTIONARY SPECTRA

Definitions

The definitions that are used in this paper are now given for both Wigner-Ville and evolutionary spectra for a nonstationary process $x(t)$. It is assumed wherever appropriate that $x(t)$ admits the required representation.

Evolutionary spectra [3]

A nonstationary process $x(t)$ is termed oscillatory if it admits the representation

$$x(t) = \int_{-\infty}^{\infty} A(t, \omega)e^{j\omega t}dX(\omega) \qquad (1)$$

where $X(\omega)$ has orthogonal increments (i.e., $E[dX^*(\omega_1)dX(\omega_2)] = 0$ for $\omega_1 \neq \omega_2$) so that

$$R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)]$$

$$= \int_{-\infty}^{\infty} A^*(t_1,\omega)A(t_2,\omega)e^{j(t_2-t_1)\omega}S_{xx}(\omega)d\omega \qquad (2)$$

It is assumed that $E[|dX(\omega)|^2] = S_{xx}(\omega)d\omega$.

The evolutionary (power) spectral density for $x(t)$ is

$$S_{xx}(t, \omega) = |A(t, \omega)|^2 S_{xx}(\omega) \qquad (3)$$

Wigner-Ville Spectra [1,2,3]

The Wigner-Ville spectrum for $x(t)$ is written

$$W_{xx}(t,\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xx}(t - \frac{\tau}{2}, t + \frac{\tau}{2})e^{-j\nu\tau}d\tau \qquad (4)$$

Relationship between the Wigner-Ville and Evolutionary Spectral Densities

From (2) and (4) it follows that

$$W_{xx}(t,\nu) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} V(t,\nu,\omega)S_{xx}(t,\omega)d\omega \qquad (5)$$

where

$$V(t, \nu, \omega) = \int_{-\infty}^{\infty} \frac{A^*(t - \frac{\tau}{2},\omega)A(t + \frac{\tau}{2},\omega)}{|A(t, \omega)|^2} e^{-j\tau(\nu-\omega)}d\tau \qquad (6)$$

Equation (6) links the two 'time-frequency' spectra and shows that the Wigner-Ville spectrum at time t and 'frequency' ν involves the evolutionary spectral density at time t and a (weighted) integral over the 'frequencies' ω, involving the function written as $V(t,\nu,\omega)$.

For the case of uniformly modulated processes, namely when $A(t, \omega) = A(t)$, then $V(t, \nu, \omega)$ may be written $V(t, \nu - \omega)$ and now V is essentially the ambiguity function [8] of the modulation $A(t)$, i.e.,

$$V(t,\nu-\omega) = \int_{-\infty}^{\infty} \frac{A(t - \frac{\tau}{2})A(t + \frac{\tau}{2})}{|A(t)|^2} e^{-j\tau(\nu-\omega)} d\tau \qquad (7)$$

Under such circumstances

$$W_{xx}(t,\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(t, \nu - \omega)S_{xx}(t, \omega)d\omega \qquad (8)$$

and $W_{xx}(t, \nu)$ is seen to be a convolution (for a fixed t) of $V(t, \omega)$ with $S_{xx}(t, \omega)$.

In order to develop the link between the two spectra more clearly a *particular form* of modulating function $A(t, \omega)$ will be selected, namely

$$A(t, \omega) = A_0(\omega)e^{-\alpha(t - T(\omega))^2} \qquad (9)$$

$A_0(\omega)$ is a time independent term and $T(\omega)$ is a 'delay' that is assumed to be frequency dependent. This choice of $A(t, \omega)$ allows the synthesis of a nonuniform modulation that resembles (in part at least) some characteristics of the Doppler shifted signals considered in the example later. This is achieved by suitable selection of the function $T(\omega)$ for each ω. The peak value of $A(t, \omega)$ occurs at $t = T(\omega)$ when $A(t, \omega) = A_0(\omega)$.

Under these circumstances

$$V(t, \nu, \omega) = \int_{-\infty}^{\infty} \frac{|A_0(\omega)|^2 e^{-\alpha(t-\frac{\tau}{2}-T(\omega))^2} e^{-\alpha(t+\frac{\tau}{2}-T(\omega))^2}}{|A_0(\omega)|^2 e^{-2\alpha(t-T(\omega))^2}} e^{-j\tau(\nu-\omega)} d\tau \quad (10)$$

Simplifying the integrand yields

$$V(t, \nu-\omega) = \int_{-\infty}^{\infty} e^{-\frac{\alpha\tau^2}{2}} e^{-j\tau(\nu-\omega)} d\tau \quad (11)$$

which is, conveniently, independent of $A_0(\omega)$ and $T(\omega)$. Now using the result

$$\int_{-\infty}^{\infty} e^{-at^2} e^{-j\Omega t} dt = \sqrt{\frac{\pi}{a}}\, e^{-\Omega^2/4a} \quad (12)$$

it follows that

$$V(t, \nu-\omega) = \sqrt{\frac{2\pi}{\alpha}}\, e^{-\frac{(\omega-\nu)^2}{2\alpha}} \quad (13)$$

so that the Wigner-Ville spectrum is now

$$W_{XX}(t, \nu) = \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} S_{XX}(t, \omega) e^{-\frac{(\omega-\nu)^2}{2\alpha}} d\omega \quad (14)$$

It is seen that a convolution relationship links $S_{XX}(t, \omega)$ to $W_{XX}(t, \nu)$. In order to obtain a more explicit link it is now necessary treat $e^{-\frac{(\omega-\nu)^2}{2\alpha}}$ as 'narrow' near $\omega = \nu$ as compared with the variation of (with ω) $S_{XX}(t, \omega)$, i.e., $e^{-\frac{(\omega-\nu)^2}{2\alpha}}$ is assumed to be a 'pseudo delta function' with respect to $S_{XX}(t, \omega)$. If $S_{XX}(t, \omega)$ is expressed as a Taylor expansion in the vicinity of $\omega = \nu$, then we can write

$$S_{XX}(t, \omega) = S_{XX}(t, \nu) + (\omega-\nu)S'_{XX}(t, \nu) + \frac{(\omega-\nu)^2}{2!} S''_{XX}(t, \nu) + \ldots \quad (15)$$

where $S'_{XX}(t, \nu)$ denotes $\left.\frac{\partial S_{XX}(t, \omega)}{\partial\omega}\right|_{\omega=\nu}$, etc.

Since $1/\sqrt{2\pi\alpha} \int_{-\infty}^{\infty} e^{-\frac{(\omega-\nu)^2}{2\alpha}} d\omega = 1$, $W_{XX}(t, \nu)$ in (14) may be written

$$W_{XX}(t, \nu) = S_{XX}(t, \nu) + S'_{XX}(t, \nu) \frac{1}{\sqrt{2\pi\alpha}} \times \int_{-\infty}^{\infty} (\omega-\nu) e^{-\frac{(\omega-\nu)^2}{2\alpha}} d\omega + \frac{S''_{XX}(t,\nu)}{2} \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} (\omega-\nu)^2 e^{-\frac{(\omega-\nu)^2}{2\alpha}} d\omega + \quad (16)$$

Neglecting terms of third and higher order we see that

$$W_{XX}(t, \nu) - S_{XX}(t, \nu) \simeq \frac{S''_{XX}(t,\nu)}{2} \cdot \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} (\omega-\nu)^2 \times e^{-\frac{(\omega-\nu)^2}{2\alpha}} d\omega \quad (17)$$

from which it is clear that the sign of $S''_{XX}(t, \nu)$ decides whether the evolutionary spectral density is greater than the Wigner-Ville spectral density, i.e.,

if $S''_{XX}(t, \nu)$ is positive $S_{XX}(t,\nu) < W_{XX}(t, \nu)$
and
if $S''_{XX}(t, \nu)$ is negative $S_{XX}(t,\nu) > W_{XX}(t, \nu)$.

It must be emphasised that these are *not* general results but have been derived using a particular form of modulating function $A(t, \omega)$. However, the implications of these results provide the background for the interpretation of the example given later.

COVARIANCE EQUIVALENT RANDOM PROCESSES AND THEIR SPECTRA

The principle of 'covariance equivalence' has been described and used elsewhere [4,5,6].

Consider a function y which is a function of some variable u, i.e., $y(u)$. Assume that $y(u)$ is a (zero mean) stationary (with respect to u) stochastic process with autocovariance function (ACVF) $E[y(u_1)y(u_2)] = R_{yy}(|u_1 - u_2|)$. To create a frequency modulated process let us regard u as a function of another variable t (time). Now let us describe y regarded as a function of time, i.e., $\tilde{y}(t) = y[u(t)]$.

Now $y(u)$ is *assumed* to have a shaping filter representation, i.e.,

$$y(u) = \underline{c}^T \underline{x}(u) \quad (18)$$

with $$d\underline{x}/du = A\underline{x}(u) + \underline{b}w(u) \quad (19)$$

A is an $(n \times n)$ constant matrix and $\underline{b}$, $\underline{c}$ are $n \times 1$ vectors; $\underline{x}$ is an $n \times 1$ state vector and w is a white process with $E[w(u_1)w(u_2)] = \delta(u_1 - u_2)$.

Our objective is to obtain a model for $\tilde{y}(t)$ and it may be shown that a process $y_1(t)$ may be generated which is '*covariance equivalent*' to $y(t)$, i.e.,

$$R_{y_1y_1}(t_1, t_2) = R_{\tilde{y}\tilde{y}}(t_1, t_2) \quad (20)$$

The process $y_1(t)$ satisfies

$$y_1(t) = \underline{c}^T \underline{x}_1(t) \quad (21)$$

with $$d\underline{x}_1/dt = \dot{u}A\underline{x}_1(t) + \sqrt{\dot{u}}\,\underline{b}w_1(t) \quad (22)$$

with $$E[w_1(t_1)w_1(t_2)] = \delta(t_1 - t_2).$$

The above may be generalised to include uniform modulation in addition to the frequency modulation by allowing the vector $\underline{c}$ to be u dependent, in which case $y(u)$ is no longer stationary with respect to u. We shall use this more general form later.

Wigner-Ville Spectra

The Wigner-Ville spectrum is easy to write down in integral form. We require $R_{\tilde{y}\tilde{y}}(t-\tau/2,\ t+\tau/2)$, and this follows directly from (18) and (19), by the usual solution of a state described system (see [7]).

Evolutionary Spectra

Evolutionary spectra cannot be obtained for y(t) but can be for $y_1(t)$. If $w_1(t)$ in (22) is (formally) expressed as

$$w_1(t) = \int_{-\infty}^{\infty} e^{j\omega t} dW(\omega) \qquad (23)$$

with power spectral density for $w_1(t)$, $S_{w_1w_1}(\omega) = 1/2\pi$, then, using (23) in (21), (22) we get

$$y_1(t) = \int_{-\infty}^{\infty} A(t,\ \omega) e^{j\omega t} dW(\omega) \qquad (24)$$

where

$$A(t,\omega) = \underline{c}^T(u(t)) \int_{\infty}^{\infty} \Phi_{\dot{u}A}(t, t-\tau)\sqrt{\dot{u}}(t-\tau) e^{-j\omega\tau} d\tau \underline{b} \qquad (25)$$

The evolutionary spectral density for $y_1(t)$ and hence by covariance equivalence $\tilde{y}$ is

$$S_{y_1y_1}(t,\omega) = \frac{|A_t(\omega)|^2}{2\pi} \qquad (26)$$

$\Phi_{\dot{u}A}$ is the state transition matrix for the system in (22).

It is noted that (25) may be written in differential equation form, i.e.,

$$A(t,\omega) = \underline{c}^T(u(t))\underline{m}(t,\ \omega) \qquad (27)$$

where vector $\underline{m}(t,\ \omega)$ satisfies

$$\dot{\underline{m}}(t,\ \omega) + [j\omega I - \dot{u}A]\underline{m}(t,\ \omega) = \sqrt{\dot{u}}\ \underline{b} \qquad (28)$$

with appropriate initial conditions.

It is noted that a first approximation to the solution of this equation (i.e., ignore $\dot{\underline{m}}$) gives a version of Grenier's [1] spectrum.

SPECTRA DUE TO A CONVECTING MONOPOLE SOURCE

As previously given in [7], Fig. 1 depicts the geometry corresponding to a convecting monopole source moving at constant velocity relative to a fixed observer. The pressure received by the observer in the far field is p(t), where

$$p(t) = \frac{s[u(t)]}{r(t)[1 - M\cos\theta(t)]^2} = C(t)s[u(t)] \qquad (29)$$

where u(t) is the so-called retarded time given by

$$u(t) = t - r(t)/c \qquad (30)$$

where M is Mach number of the source, c is the speed of sound and r(t) the distance from source to observer, at the time when the signal received was generated.

s(u) is the monopole source strength and is assumed 'stationary' in u with spectral density having a single peak in form that allows a state representation [7]. The functional form for u(t) is fixed by the (constant) speed of the monopole and the 'flyover' geometry. The required state parameters (e.g., transition matrix) are available in analytic form. This provides sufficient information for computation of both Wigner-Ville and evolutionary spectra for p(t).

The results of the computation are shown in Figs. 2-4. Figure 2 is the evolutionary spectral density (which is very similar in form to the Wigner-Ville spectrum). To accentuate the difference the relative difference is computed, i.e.,

$$\frac{S_{xx}(t,\ \omega) - W_{xx}(t,\ \nu)}{S_{xx}(t,\ \omega)}$$

for $\omega = \nu$, in Figs. 3, 4.

The discussion presented earlier does not apply *directly* here, owing to the fact that the modulating function is not the relatively simple form used before. However, it is clear from Figs. 3 and 4 that $S_{xx}(t,\ \omega) < W_{xx}(t,\ \omega)$ in one region and $S_{xx}(t,\ \omega) > W_{xx}(t,\ \omega)$ in another (for a fixed t as ω varies) and this does correspond to changes in S''_{xx} *broadly* as discussed before.

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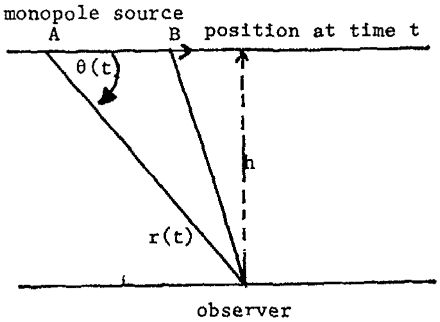

signal is received by observer at time t due to source when at position A at time $t-\frac{r(t)}{c}$

Fig. 1 'Flyover' geometry.

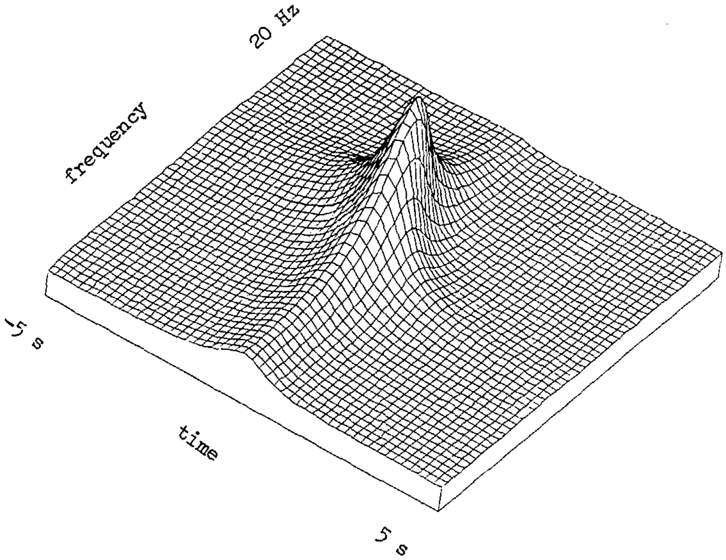

Fig. 2. The evolutionary spectral density.

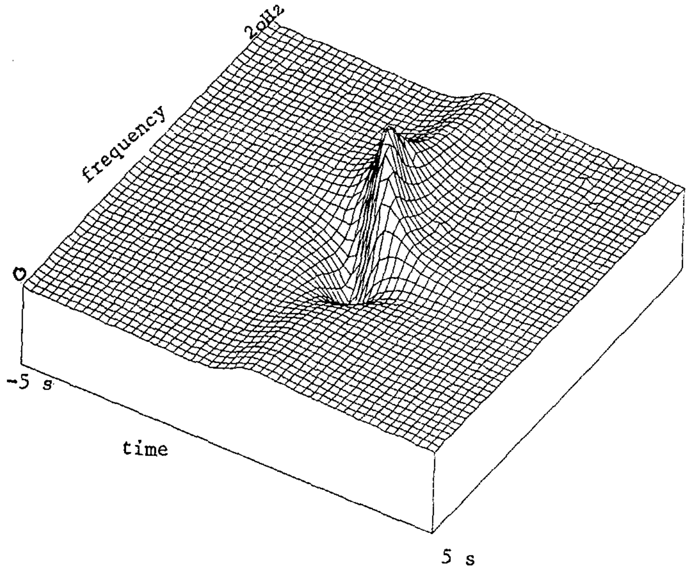

Fig. 3 Relative difference between evolutionary and Wigner-Ville spectrum

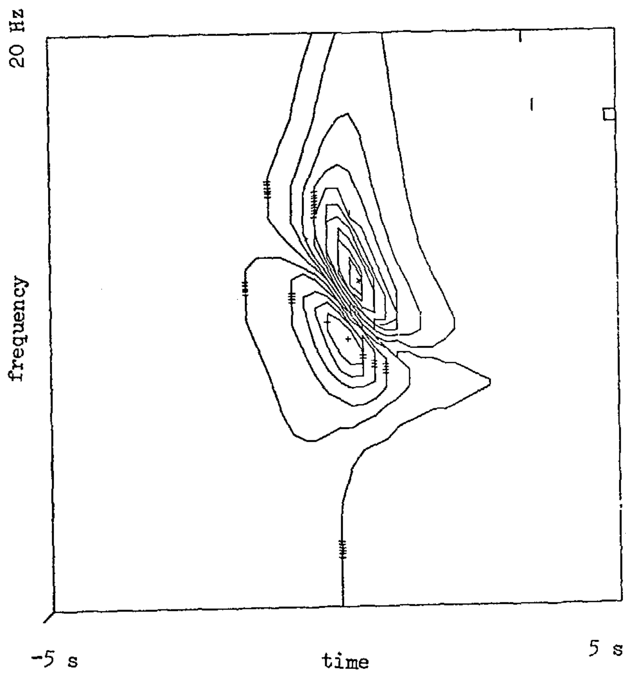

Fig. 4. Relative difference as a contour plot.