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ASYMPTOTIC BEHAVIOUR AND STATICTICS OF SPECTRAL MOMENTS

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RESUME

Dans cette étude on a consideré l'éstimation des moments spectraux avec ses applications. Les moments spectraux sont fréquemment utilisés dans l'anémometrie, mesurage de type Doppler etc., où les premiers trois moments correspondent, respectivement, au volume, à la vitesse moyenne et à la bande passante des diffracteurs. La téchnique d'éstimation utilisée est une expansion en série des termes d'autocorrelation. On peut faire usage de la méthode d'entropie maximum pour extrapoler la serie d'autocorrelation.

En particuliere nous avons analyse le cas de sinusoïdes contenus dans un bruit blanc additif. En fait dan ce cas les caracteristiques statistiques et asymtotiques sont explorés. Le problème de test d'hypothèse est aussi formule en utilisant les moments.

SUMMARY

In this work the problems of the estimation of spectral moments and their applications have been addressed.

The spectral moments are frequently used in anemomentry, Doppler measurements etc., where the first three moments correspond respectively to the volume, mean velocity and range of velocities of the scatterers. The estimation technique used is a series expansion in terms of the autocorrelation lags. Use can be made of the maximum entropy techniques to extrapolate the autocorrelation lags from a few known or estimated lags. Special attention is given to the case of sinusolds in additive white noise. Infact for this case, properties, statistics and asymptotic behaviour of the moments have been investigated. The moment estimates are also used for the hypothesis testing problem where the alternative and null hypotheses, represent, respectively, the tone in noise and noise only situation. The potential use of the moments for the tone frequency estimation is also considered.

1. INTRODUCTION

Spectral moments have been used in various applications in radar and sonar problems [1], tone detection [2], mean frequency estimation [3], and convolution, deconvolution problems [4]. As an example, in radar applications the first three moments correspond, respectively, to the volume, mean velocity and range of velocities of the scatterers. The spectral moments are conventionally computed by first obtaining an estimate of the spectrum function S(f) and then using the moment integrals. One can however estimate the spectral moments using time domain information as well, i.e., by using the inphase and quadrature signals [2] or by using a series expansion in terms of the autocorrelation lags as in [5].

In this study we investigate the use of maximum entropy spectrum in the estimation of moments and consider some of their elementary properties. In the lag expansion method, for the moments we consider and compare the asymptotic behaviour of the estimator with the case when only a few lag terms are involved. We obtain an expression for the probability density function of the n'th moment and expand on its potential use.

2. MESA MOMENTS

Let us consider as an approximation the the moment integrals the sum in (1) with the N-point discrete Fourier transform :

$$m_n = \frac{2}{N} \sum_{\ell=0}^{N-1} S(\ell) \left[\frac{\ell T}{N}\right]^n \qquad (1)$$

which is valid when T/N is small relative to variations in the spectrum and where m_n and T denote, respectively, the n'th moment and the sampling period. Expanding the factor (T/N) interms of a cosine series [5], one obtains a moment expression in terms of the autocorrelation (AC) lags, i.e.,

$$m_n = \frac{c_o^n r_o}{2} + \sum_{k=1}^{\infty} c_k^n r_k \qquad (2)$$

where

$$c_k^n = 4T \left[(-1)^k \sum_{\ell=0}^{n-1} (-1)^r \frac{n!}{(n-fr-1)} \frac{(2T)^{-n+2r+1}}{(2\pi kT)^{2r+2}} \right.$$

$$\left. \begin{cases} (-1)^{(n-1)/2} \dfrac{n!}{(2\pi kT)^{n+1}} & n = 1,3,5,\ldots \\ 0 & n = 0,2,4,\ldots \end{cases} \right]$$

and

$$r_k = \frac{2}{N} \sum_{\ell=0}^{N-1} S(\ell) \cos\left(\frac{2\pi k\ell}{N}\right) \quad k = 0,1,..(N-1)$$

In practice only the first M AC terms may be available; however higher order lags can be found by extrapolation using the equation :

$$r_k = -a_1 r_{k-1} - a_2 r_{k-2} \ldots\ldots a_M r_{k-M} \quad k > M \qquad (3)$$

where the $\{a_j$, $j=1,2,\ldots M\}$ are the linear prediction coefficients found by one of the least squares (LS) techniques such as, PARCOR, sequential LS, lattice, covariance, autocorrelation ... algorithms. Because of the potential implication of the maximum entropy techniques in (3), we will call the estimates in (1) the MESA moments.

2.1 Elementary Properties

Certain properties of spectral moments obtained through the expansion in (2) are listed below.

-Shifting : The shifted moments

$\mu_n = \int_0^{\infty} (f-f_o)^n S(f)df$ can be found as

$$\mu_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} m_{n-i} f_o^i \qquad (4)$$

-Filtering : A process filtered through a system with power transfer function P(f) has the moments

$$\mu_n = \int_0^{\infty} f^n S(f)p(f)df \cong \frac{C_o^n}{2} r_o' + \sum_{k=1}^{\infty} C_k^n r_k' \qquad (5)$$

where

$$r_k' = \sum_{i=0}^{N-1} r_i \lambda_{k-i} \quad \text{and} \quad \lambda_k = \sum_{\ell=0}^{N-1} P(\ell) \exp(j\frac{2\pi}{N}k\ell)$$

-Windowing : If the process is windowed with the window function w(n), the moments take the form

$$\mu_n = \frac{C_o^n r_o'}{2} + \sum_{k=1}^{\infty} C_k^n r_k' \qquad (6)$$

with

$$r_k' = (w(k) * w(k))\, r_k$$

2.2 Asymptotic behaviour

It is of interest to evaluate and compare the moment estimates in (1) increasing number of AC terms. Since the case of tones in the noise background is analytically tractable let us infact consider their asymptotic behaviour. The AC sequence for L tones in white noise is given by

$$r_k = \sigma_n^2 \,\delta(k) + \sum_{i=1}^{L} A_i^2 \cos w_i k \qquad (7)$$

where σ_n^2 is the noise variance and $\delta(k)$ is the Kronecker delta.

For the noise free case ($\sigma_n^2 = 0$) and a single tone ($L=1$), the first moment is given by

$$m_1 = \frac{A_1^2}{4T} - \frac{2A_1^2}{\pi^2 T} \sum_{k=1}^{\infty} \frac{\cos w_1(2k-1)}{(2k-1)^2} \qquad (8)$$

Here using [6,pp.39] and considering the fact that m_1/m_o and $[(m_2-m_1^2)/m_o]^{1/2}$ are estimates, respectively, of the mean frequency and mean square bandwidth (BW) one obtains the exact values $\hat{f}=f_1$ and $\hat{BW}=0$. For the case of $\sigma_n^2 \neq 0$, the mean frequency and bandwidth become, respectively,

$$\hat{f} = \frac{\mu}{\mu+1} \left[\frac{1}{4\mu T} + f_1\right] \qquad (9\text{-a})$$

$$BW = \left[\frac{\mu+1}{\mu^2} \left(f_1^2 + \frac{4\mu+1}{48\,\mu T^2} - \frac{f_1}{2T}\right)\right]^{\frac{1}{2}} \qquad (9\text{-b})$$

where $\mu = A_1^2/\sigma_n^2$ is the signal to noise ratio.

In Equation (9-a) the term $\frac{1}{4T}$ is due to the white noise mass; this bias however be eliminated and this equation can be used as a tone frequency estimator. The convergence properties of the series in (2) is shown in Fig.1 where both $\hat{f}$ and BW^2 are plotted as functions of the signal to noise ratio (SNR) and parametrically dependent upon the number of lag terms M. In this example one has T=0.05 sec, hence the Nyquist frequency is 10 Hz and the tone frequency is 1 Hz as in [7]. For SNR→0, one expects $\hat{f} = \frac{1}{4T}$ and $BW = \frac{1}{2T}$ while for SNR→∞ one should have $\hat{f}$=1 Hz. One observes also that a good estimate of the moments can be obtained by using only a few AC lags, i.e., $5 < M < 10$. The first and second moments can be reliably estimated using two lags, however higher order moments require more than two lags.

Two-tone case : The first moment expression for two tones in the noise-free and noisy cases become respectively :

$$m_1\Big|_{\sigma_n^2=0} = A_1^2 f_1 + A_2^2 f_2 \qquad (10\text{-a})$$

and

$$m_1\Big|_{\sigma_n^2\neq 0} = A_1^2 f_1 + A_2^2 f_2 + \frac{\sigma_n^2}{4T} \qquad (10\text{-b})$$

Here m_1/m_o yields the power weighted tone frequencies.

2.3 Statistical Properties

It is of interest to consider the probability density function (p.d.f) of the n'th moment under the tone present (H_1) and noise only (H_o) hypotheses. We assume that the number of samples used N is large enough to assume Gaussian statistics. Let us consider again a two term approximation, for the moments, i.e.,

$$m_n = \frac{c_o^n r_o}{2} + c_1^n r_1 \qquad (11)$$

Statistics for m_n involving an arbitrary number of AC lags can be obtained in a straightforward but tedious manner.

For both the null and the alternative hypotheses, m_n has a Gaussian distribution with mean and variance, respectively as [8,12]

$$\lambda_{0,n} = E\left(m_n \mid H_o\right) = \frac{a_n \sigma_n^2}{2} \qquad (12)$$

$$\lambda_{1,n} = E\left(m_n \mid H_1\right) = \frac{a_n}{2}\left(\sigma_n^2 + A_1^2\right) + b_n A_1^2 \cos w_1 \qquad (13)$$

$$\sigma_{0,n}^2 = E\left[\left(m_n - \lambda_{0,n}\right)^2\right] = \frac{a_n^2 \sigma_n^4}{4N} + \frac{b_n^2 \sigma_n^4}{N} \qquad (14)$$

$$\sigma_{1,n}^2 = E\left[\left(m_n - \lambda_{1,n}\right)^2\right] = \frac{\sigma_n^2}{N}\left[\beta\left(a_n^2 + \frac{b_n^2}{2}\right) + b_n^2 A_1^2 \cos 2w_1 - d_n A_1^2 \cos w_1\right] \qquad (15)$$

where

$$\beta = \sigma_n^2 + 2A_1^2$$

$$d_n = 2c_o^n c_1^n$$

$$a_n = c_o^n$$

$$b_n = c_1^n$$

and E(.) is the expectation operator.

Let us now discuss two applications.

Tone Detection

The p.d.f. expressions for moments can be used to set the likelihood ratio. Infact for M=2 one has the log-likelihood ratio of the n'th moments [8,9] :

$$Lm_n = m_n\left[0,5\left(\frac{1}{\sigma_{0,n}^2} - \frac{1}{\sigma_{1,n}^2}\right) m_n + \left(\frac{\lambda_{0,n}}{\sigma_{0,n}^2} - \frac{\lambda_{1,n}}{\sigma_{1,n}^2}\right)\right] \underset{H_o}{\overset{H_1}{\gtrless}} m_t \qquad (16)$$

with

$$m_T = \frac{\lambda_{1,n}^2}{2\sigma_{1,n}^2} - \frac{\lambda_{0,n}^2}{2\sigma_{0,n}^2} + \ln\zeta - \ln\frac{\sigma_{1,n}}{\sigma_{0,n}}$$

being the threshold of the test. The false alarm and detection probabilities are, of course, given by

$$P_F = Q\left(\frac{m_T - \lambda_{0,n}}{\sigma_{0,n}}\right) \quad \text{and} \quad P_D = Q\left(\frac{m_T - \lambda_{1,n}}{\sigma_{1,n}}\right) \qquad (17)$$

where Q(.) denotes the error function.

It will be of interest to compute the efficacies of the above test and to compare the hypothesis tests on the n'th moment (n=1,2,.....) based on their asyptotic relative efficiency (A.R.E.) figures.

Mean Frequency Estimation

The first moment can be used to estimate the mean frequency as follows :

$$\bar{f} = \frac{0,25}{T} - \frac{0.2}{T}\sum_{k=1}^{m/2} \frac{r_{2k-1}}{r_o}\frac{1}{(2k-1)^2} \qquad (18)$$

In Fig.2 the estimated mean frequency is plotted versus the number of known AC lags. (Extrapolation goes from 3 to 40 starting with the two known AC lags.) The simulated tone frequency is 1 Hz and sampling rate 20 Hz. One observes an oscillatory behaviour for $\bar{f}$ up to M=30, however higher order AC terms can be obtained by extrapolation through Eq.3. The phase dependency of MESA mean frequency estimator is shown in Fig.3. It can be observed that the error depends on the initial phase of the tone. For example for $6 < M < 8$ minimum frequency error is attained at about $\phi=180^o$ while for M=18, good performance is attained, at $\phi=90^o$ and 210^o. Also the values of the estimated frequency using the known AC are straight lines as shown in Fig. 3.

Variance of the mean frequency :

A measure of the accuracy of the mean frequency estimate the MESA method is given by the variance analysis. In the calculation of the variance of $\bar{f}$ use has been made of the variance expressions for the AC terms as below :

The variance of the unbiased estimate of the i'th AC terms is [10]

$$Var\{r_i\} = \frac{1}{(N-i)^2}\sum_{k=-(N-i)}^{(N-i)} (N-i-k \quad)$$

$$(r_k^2 + r_{k-i}\, r_{k-i}) - E^2(r_i) \qquad (19)$$

Using second order approximation technique as in [11] and assuming that the error in estimating r_{2k-1} is uncorrelated with r_o, we have finally

$$V_{ar}\,\hat{f} = \sigma^2_{\hat{f}} = 0.04\ T^{-2}\ V_{ar} \sum_{k=1}^{m/2} \frac{\Gamma_{2k-1}}{(2k-1)^2} \qquad (20)$$

where $\Gamma_{2k-1} = \dfrac{r_{2k-1}}{r_o}$

Let us again specialize to the case of a sinusoidal signal in white noise and consider two situations seperately for analytical simplicity. The variance of the mean frequency (N), sampling period and mean frequency [8]

Noise free ($\sigma^2_n = 0$)

$$\sigma^2_{\hat{f}} = 0.04\ T^{-2} \left[(\beta_k + 2h_{jk}) + \frac{f(k)}{N} \left(1 + \frac{2\,g(j)}{N}\right) \right] \qquad (21)$$

Noisy case ($\sigma^2_n \neq 0$)

$$\sigma^2_{\hat{f}} = \frac{0.04\ T^{-2}\ \mu^2\ f(k)}{(1+\mu)^4\ N} \left[\mu^2 + \frac{2\mu^2}{N}\, g(j) + (1+4\mu) \right] +$$

$$\frac{0.04\ T^{-2}}{(1+\mu)^2} \left[\mu^2 \beta_k + 2\mu^2\, h_{jk} + \beta_k + 2\mu\, (\beta_k + \xi_k) \right] \qquad (22)$$

where we have used the definitions :

$$h_{jk} = \sum_{k=1}^{M/2} \sum_{j=1}^{M/2} \frac{(N-2k+1-j)}{(N-2k+1)^2}\ \frac{\cos 2jw_1}{(2k-1)^2}$$

$$\xi_k = \sum_{k=1}^{M/2} \frac{(N-4k+2)\cos(4k-2)\,w_1}{(n-2k+1)^2 (2k-1)^2}$$

$$\beta_k = \sum_{k=1}^{M/2} \frac{1}{(N-2k+1)(2k-1)^2}$$

$$f(k) = \sum_{k=1}^{M/2} \frac{\cos^2(2k-1)\,w_1}{(2k-1)^2}$$

$$g(j) = \sum_{j=1}^{N} (N-j) \cos 2jw_1$$

The variation of the mean frequency estimation variance with respect to the number of AC terms as well as with respect to the μ is shown in Figs.4 and 5 respectively. It can be observed from Fig.4, the variance of the mean frequency estimate does not decrease with increasing M. For fixed N, the AC estimation errors at higher order lags account for this behaviour. The saturation in Fig.4 is due to the k^{-2} term Eq.(20).

CONCLUSIONS

In this study we investigate the use of MESA method in the estimation of spectral moments and consider some of their elementary properties. A good estimate of the moments can be obtained by using oly a few AC lags i.e. $5 < M < 10$. The first and second moments can be reliably estimated using two lags, however higher order moments require more than two lags.

The analysis of statistical properties reveals that P_F and P_D can be written as functions of the expected value and the variance of n'th moment for the tone detection problem. It will be of interest to compute the hypothesis tests on the n'th moment (n=1,2...) based on their asymptotic relative efficiency.

Simulation studies have shown that after a certain value of M (typically 10 to 20) there is a little improvement in the tone frequency estimate.

REFERENCES

1)- D.S. Zrnic "Estimation of Spectral Moments for Weather Echoes" IEEE, Trans. Geosc. Elect., GE-17, pp 112-128, October 1979.

2)- J.N. Denenberg "Spectral Moment Estimators: A new Approach to Tone Detection", The Bell system Technical Journal, Vol.55, pp 143-155, February 1976.

3)- K.S. Miller. "A Covarianca Approach to Spectral Moment Estimation", IEEE, Trans. Infor. Theory, IT-18, pp 588-597, September 1972.

4)- E. Sjontoft "Convolution and Deconvolution When Only The Lower Order Moments of the Convolution Function are known". GRETSI, Proc., pp 282-283,Nice 1981.

5)- S. Holm "Spectral Moment Matching in the MESA Method "IEEE, Trans, Infor. Theory, IT-29,pp 311-313,March 1983

6)- I.S. Gradsteyn, Table of Integrals, Series and Product Relations, Acedemic Press, New-York, 1965.

7)- W.Y. Chen and G.R. Stegen "Experiment with Maximum Entropy Power Spectra of Sinusolds "J.Geophys, Res., vol. 79, July 1974.

8)- E. Anarın "Improvements and Investigations on Modern Spectrum Estimation", Ph.D Thesis in preparation, Dept. of Elec. Eng, Boğaziçi University, 1985.

9)- H.V. Trees, Detection, Estimation and Modulation Theory, J. Wiley, New-York 1968.

10)- G. Box and G.M. Jenkings, Time Series Analysis : Forecasting and Control, Holden Day, San Francisco, 1970.

11)- A. Padoulis, Probability Random Variables and Stochastic Processes, M. Graw Hill, New-York, 1965.

12)- S.M. Kay "Robust Detection by Autoregressive Spectrum Analysis, IEEE Trans ASSP. Vol. ASSP-30, No.2 pp 256-269, April 1982.

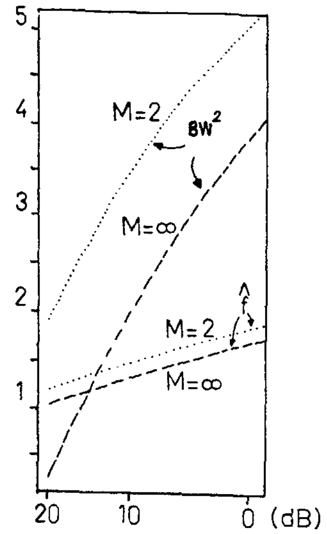

Fig-1. Variations of $\hat{f}$ and BW^2 with 10 Log (μ)

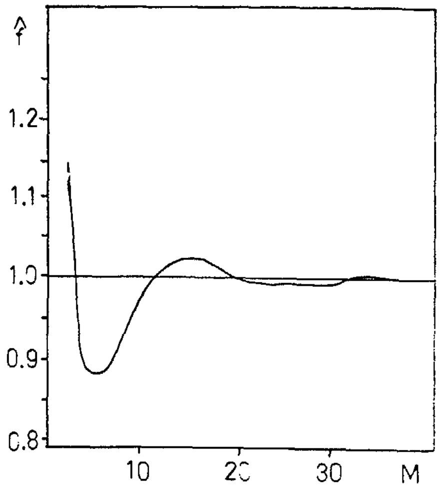

Fig-2. Estimation tone frequency plotted parametrically versus number of AC terms.

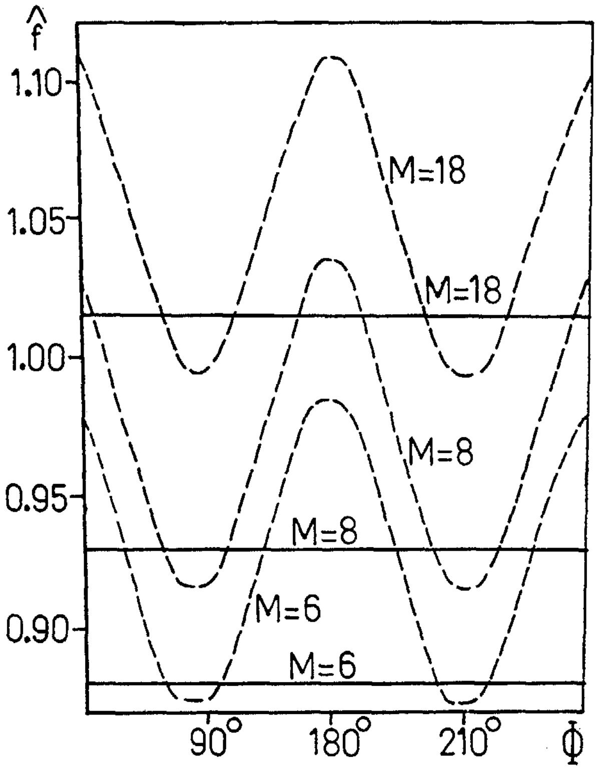

Fig-3. Variation of the mean frequency estimation with the initial phase (N=100)

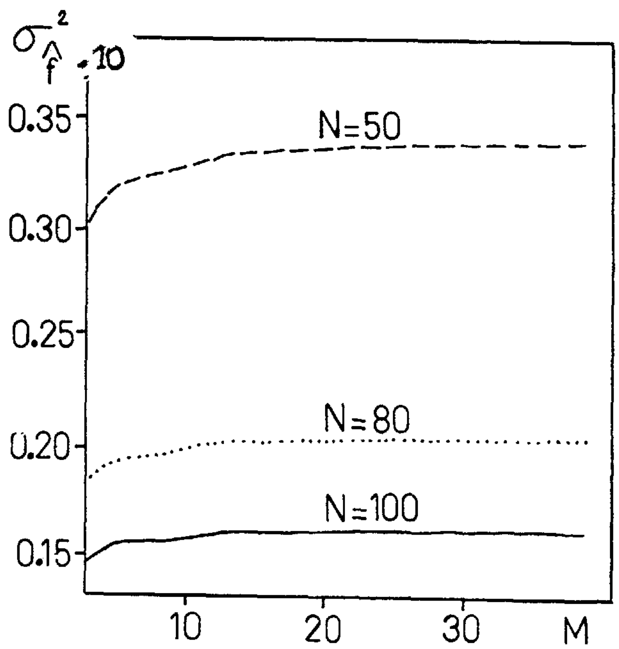

Fig- 4. $\sigma^2_{\hat{f}}$ versus number of AC terms.

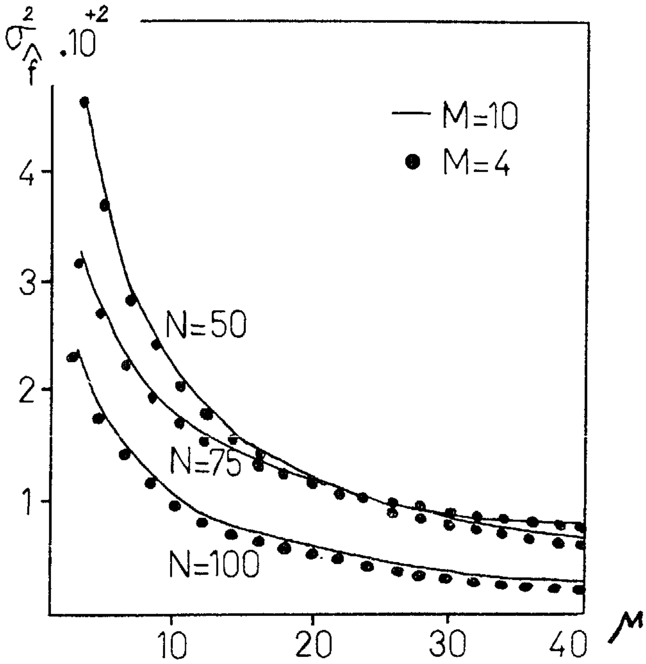

Fig-2. Variation of the mean frequency estimation variance with respect to $\mu = A_1^2/\sigma_n^2$