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# APPLICATIONS OF STOCHASTIC INTEGRAL EQUATIONS TO WAVE PROPAGATION IN A RANDOM MEDIUM

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**RESUME**

Ce travail est une introduction à la théorie des systèmes spatiales stochastiques, inspirée par la méthode des équations intégrales stochastiques.

La technique employée a été développée par K.C. LIU pour analyser la propagation des ondes acoustiques sous-marines.

Nous commençons avec une section présentant les définitions et les notations nécessaires. Ensuite le problème de la propagation des ondes dans un milieu aléatoire est consideré.

Enfin nous étudions le cas de la propagation des ondes radioélectriques par diffusion provenant de la troposphère.

**SUMMARY**

This paper is an introduction to the theory of spatial stochastic systems, and the approach used is inspired by the method of stochastic integral equations.

The technique used has been developed by K.C. LIU to analyse wave propagation problems in underwater acoustics.

We start with some definitions and notations and then we consider the problem of wave propagation in a random medium.

Finally we extend some previous results to the case of troposcatter propagation of radio waves.

# APPLICATIONS OF STOCHASTIC INTEGRAL EQUATIONS TO WAVE PROPAGATION IN A RANDOM MEDIUM

## 1. INTRODUCTION

Signals transmitted through a random communication channel and signals scattered back as target echoes and reverberation from volume and boundaries suffer random variations in time and space. The transmitting medium including its random boundaries can be characterized as a linear time and space variant random filter.

Wave propagation in a random medium may be described with the aid of linear spatial stochastic systems theory. The relation between source excitation and resulting field may be expressed using a stochastic system operator.

A spatial stochastic system may be resolved into suitable subsystems to decompose a complex physical process into several simpler processes.

This method has been developed by K.C. LIU [1] to analyse wave propagation problems in underwater acoustics. The technique can be applied as well to optical and electromagnetic problems as for example troposcatter propagation of radio waves.

## 2. DEFINITIONS

A linear spatial stochastic system operator describes a stochastic linear transformation of a random input source excitation function s into a random output field function f.

$$f(\vec{y},\xi) = \Omega\{s(\vec{x},\xi)\} \quad . \qquad (1)$$

The source excitation function s is a function of a vector $\vec{x} \in R^n$ and a stochastic ensemble parameter $\xi$ from the space of elementary events. The resulting field function f is a function of a vector $\vec{y} \in R^m$ and $\xi$. Linearity is equivalent to the validity of the superposition principle which implies the independent propagation of waves. The dimensions n of the vector $\vec{x}$ and m of the vector $\vec{y}$ may vary from one to four, with one time dimension and zero, one, two or three physical space dimensions.

If m is smaller than n, then the system operator omega is called compressive, if m is larger than n, then omega is called extensive and if m is equal to n then omega is called homospatial.

If one defines in the case of a linear receiving array the incident wave as input source function s and the electrical output of the array as output field function f then the system operator will be compressive with n=4 and m=1.

If one defines in the case of ambient noise caused by the sea surface the random noise source caused by the sea waves as input source functions s and the resulting noise field as output field function f then the system operator will be extensive with n=3 and m=4. If one defines in the case of random volume scattering the incident wave as input source function s and the scattered wavefield as output field function f then the spatial stochastic system operator will be homospatial and n=m=4.

## 3. SPATIAL STOCHASTIC SYSTEM FUNCTIONS

The stochastic field caused by a Dirac delta function in the n-dimensional source space is called the unit impulse response function of the stochastic system.

$$g(\vec{y},\vec{x}_0,\xi) = \Omega\{\delta(\vec{x}-\vec{x}_0)\} \quad . \qquad (2)$$

Using the sifting property of the Dirac delta function we can decompose an arbitrary source function s into a linear combination of weighted and displaced delta functions.

$$s(\vec{x},\xi) = \int_{R^n} s(\vec{x}_0,\xi)\,\delta(\vec{x}-\vec{x}_0)\,d\vec{x}_0 \quad . \qquad (3)$$

To find the resulting field function f of an arbitrary source functions s we substitute (3) into (1)

$$f(\vec{y},\xi) = \Omega\Big\{\int_{R^n} s(\vec{x}_0,\xi)\,\delta(\vec{x}-\vec{x}_0)\,d\vec{x}_0\Big\}. \qquad (4)$$

Regarding $s(\vec{x}_0,\xi)$ as a weighting factor applied to the elementary function $\delta(\vec{x}-\vec{x}_0)$ we can use the linearity property of the stochastic system operator to allow omega to operate on the individual elementary func-

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tions. Thus we get

$$f(\vec{y},\xi)= \int_{R^n} s(\vec{x},\xi)\Omega\{\delta(\vec{x}-\vec{x}_o)\}dx_o \quad . \qquad (5)$$

As a final step we use the definition of the impulse response function given in equation (2) and we get the superposition integral

$$f(\vec{x},\xi)= \int_{R^n} s(\vec{x}_o,\xi)g(\vec{y},\vec{x}_o,\xi)d\vec{x}_o \quad . \qquad (6)$$

The transfer function of the stochastic system is given as a n-dimensional Fourier-transform of the impulse response function into the wave number space.

The second order moments may be calculated as the mathematical expectations with respect to the stochastic ensemble parameter $\xi$. There exists a Fourier-relationship between the second order moments of the impulse response function and of the transfer function similar to the Wiener-Khinchin or to the Van Zittert-Zernicke theorem.

In many cases second order moment theory is only used for convenience. To give a complete description one should know the multidimensional probability distributions of the random field functions.

## 4. STOCHASTIC INTEGRAL EQUATIONS

K.C. LIU proposed a method to determine the transfer functions of complex spatial stochastic systems by resolving them into several simpler systems [1]. If impulse responses or transfer functions of the simpler physical processes that correspond to these basic systems are known from theory or measurements then the calculation of the complex process can be transformed to the problem of solving stochastic integral equations.

Thus the problem of solving partial differential equations with randomly moving and rough boundaries may be transformed into a problem of solving integral equations. There are four types of basic connections. The first type is a parallel network of spatial stochastic systems. Their unit impulse response functions add up to form a total impulse response.

$$G_T(\vec{y},\vec{x},\xi) = \sum_{j=1}^{n} g_j(\vec{y},\vec{x},\xi) \qquad (7)$$

The second type is a cascade connection of homospatial subsystems. The unit impulse response of the total system is given by repeated convolution of the individual impulse respones of the subsystems:

$$G_T(\vec{x}_n,\vec{x}_o,\xi) = \underbrace{\int\dots\int}_{n-1} \prod^{n} g_j(x_j,x_{j-1},\xi)dx_1\dots dx_{n-1} \quad . \qquad (8)$$

The third type of basic connections is a homospatial self-feedback system $\Phi$. One gets, with I, the identity operator, $\Psi_F$ the feedforward and $\Psi_B$ the feed-back operator, the operator equation:

$$\Phi = (I-\Psi_F\Psi_B)^{-1}\Psi_F \quad . \qquad (9)$$

The fourth type is a mutual feedback connection. Such a system is a network model for multiple scattering between two boundaries, a problem which is related to acoustic wave propagation in shallow water. With $\Phi_1$ the spatial stochastic operator for the surface and $\Phi_2$ the spatial stochastic operator for the bottom and I the identity operator, we get for the total operator the mutual feedback operator equation

$$\Phi = (I-\Phi_1\Phi_2)^{-1}(\Phi_1+\Phi_1\Phi_2) + (I-\Phi_2\Phi_1)^{-1}(\Phi_2+\Phi_2\Phi_1) \qquad (10)$$

with $\varphi_1$ and $\varphi_2$ the impulse responses of the systems $\Phi_1$ and $\Phi_2$ and $s(\vec{x})$ the source excitation function we get for the resulting field function $f(\vec{x})$

$$f(\vec{x}) = f_1(\vec{x}) + f_2(\vec{x}) \qquad (11)$$

where $f_1(\vec{x})$ and $f_2(\vec{x})$ satisfy the following stochastic nonhomogeneous FREDHOLM integral equations of the second kind:

$$f_j(\vec{x}) - \alpha_j(\vec{x}) = \int \gamma_i(\vec{x},\vec{x}')f_j(\vec{x}')d\vec{x}' \qquad j=1,2 \qquad (12)$$

with the abbreviations

$$\gamma_1 = \int \varphi_1(\vec{x}-\vec{x}'',\vec{x})\varphi_2(\vec{x}''-\vec{x}',\vec{x}'')d\vec{x}''$$
$$\gamma_2 = \int \varphi_2(\vec{x}-\vec{x}'',\vec{x})\varphi_1(\vec{x}''-\vec{x}',\vec{x}'')d\vec{x}'' \quad (13)$$

$$\alpha_j(\vec{x}) = \int s(\vec{x})[\varphi_j(\vec{x}-\vec{x}',\vec{x})+\gamma_j(\vec{x},\vec{x}')]d\vec{x}' \quad (14)$$

$j=1,2$

## 5. RANDOM COMMUNICATION CHANNEL MODEL

If the propagation is established by single scattering from a large number of independent elements the WSSUS (wide sense stationary uncorrelated scattering) channel model can be applied. The scattering function $L(\mu,\nu)$ gives the distribution of the elementary point scatters or blobs in DOPPLER-shift $\mu$ and time delay $\nu$. The WOODWARD ambiguity-function is the crosscorrelation function of the transmitted complex signal $x(t)$ with the received echo, that has suffered DOPPLER spread $\mu$ and time delay .

$$\chi(\mu,\nu) = \int_{-\infty}^{\infty} x^*(t)\cdot x(t+\nu)e^{j2\pi\mu t}dt = \int_{-\infty}^{\infty} X^*(f)\cdot X(f+\mu)e^{j2\pi f\nu}df \quad (15)$$

The output of the random channel can be described as the two-dimensional convolution of the input and the scattering function of the channel [2,3]:

$$\langle|\chi_0(\phi,\tau)|^2\rangle = \iint|\chi_i(\phi-\mu,\tau-\nu)|^2 L(\mu,\nu)d\mu d\nu \quad .(16)$$

This relation may be generalized to include angular spread u and v [4]. The variables u and v are defined as the wavenumer components in the angular space

$$u = \frac{\alpha}{\lambda} \quad ; \quad v = \frac{\beta}{\lambda} \quad (17)$$

$\lambda$ denotes the wave length and $\alpha$ and $\beta$ are the direction cosines of the wavenumber vector $\vec{k}$ with respect to the x and y axis. We get the generalized input ambiguity-function as the product of the power directivity function $|\chi(u,v)|^2$ of an array and the input signal ambiguity-function $|\chi(\mu,\nu)|^2$. The fourdimensional output ambiguity-function of the random WSSUS-channel is obtained from convolution with the fourdimensional scattering function

$$\langle|\chi_0(\phi,\tau,u,v)|^2\rangle = \iiiint|\chi_i(\phi-\mu,\tau-\nu,u-\eta,v-\Psi)|^2\cdot L(\mu,\nu,\eta,\Psi)d\mu d\nu d\eta d\Psi \quad . \quad (18)$$

It gives the angular spread and the dispersion in time and frequency caused by the random medium. The Fourier-transform of the scattering function yields the fourdimensional correlation function of the transfer function of the medium:

$$\langle H^*(t,f,x,y)\cdot H(t+\Delta t,f+\Delta f,x+\Delta x,y+\Delta y)\rangle = R_H(\Delta t,\Delta f,\Delta x,\Delta y) = \iiiint L(\mu,\tau,u,v)e^{-j2\pi(\Delta t\mu+\Delta f\tau+\Delta xu+\Delta yv)}d\mu d\tau du dv \quad (19)$$

## 6. REVERBERATION

Equation (16) and its generalization (18) may be applied to the problem of clutter and reverberation. Reverberation is assumed to arise from a collection of scatterers having a random distribution in range and velocity, giving rise to an echo having a random distribution in delay and Doppler shift. The scattering function L determines how the reverberation energy in average will be distributed in delay, Doppler and angular space. The range, Doppler and angular resolution of a system in a random inhomogeneous medium can be expressed by the output ambiguity-function, expressing the combined effect of the signal and the medium. If the scattering function can be factorized

$$L(\mu,\tau,u,v) = L(\mu,\tau)\cdot L(u,v) \quad , \quad (20)$$

the corresponding correlation function of the medium-transfer function can be factorized as well

$$R_H(\Delta t,\Delta f,\Delta x,\Delta y) = R_H(\Delta t,\Delta f)\cdot R_H(u,v) \quad . \quad (21)$$

The mean reverberation Doppler shift may be calculated as

$$\bar{\mu} = \frac{1}{L_v}\iint L(\mu,\tau)d\mu d\tau \quad (22)$$

and the mean squared Doppler spread

$$\overline{\mu}^2 = \frac{1}{L_v} \iint \mu^2 L(\mu,\tau)\, d\mu\, d\tau \qquad (23)$$

with the total volume of the scattering function

$$L_v = \iint L(\mu,\tau)\, d\mu\, d\tau \;.$$

The mean delay and the mean angular spread may be defined in the same way. If we assume a Gaussian shaped scattering function

$$L(\mu,\tau,u,v) = e^{-\pi(\mu^2/a^2+\tau^2/b^2+u^2/c^2+v^2/d^2)} \qquad (24)$$

the corresponding correlation function of the transfer function of the medium will be also Gaussian:

$$R_H(\Delta t,\Delta f,\Delta x,\Delta y) = \\ = a\cdot b\cdot c\cdot d\cdot e^{-\pi(a^2\Delta t+b^2\Delta f+c^2\Delta x+d^2\Delta y)} \;. \qquad (25)$$

The assumption of wide-sense-stationary uncorrelated scattering, however, may not be valid in reality. Then more general expressions for the space-time correlation function of the scattered field have to be used as has been shown in Ref. [1].

## 7. MULTIPLE SCATTERING

We consider n randomly moving discrete scatterers in an inhomogeneous medium and an electrical signal s(t) acting on a radiator R which produces a field $\Psi(\vec{x})$. If the j-th scatterer causes a scattered field $f_j(\vec{y})$ and if its corresponding system operator is $\Phi_j$, then the total spatial stochastic system is given according to [1] as:

$$\Phi = \left\{ \sum_{i=1}^{n} \Phi_i + \sum_{i=1}^{n} \sum_{\substack{j=1\\ i\neq j}}^{n} \Phi_i \Phi_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\substack{k=1\\ i\neq j\,;\, j\neq k}}^{n} \Phi_i \Phi_j \Phi_k + \dots \right\} R \qquad (26)$$

The first term gives the contribution of first scattering, the second term that of the second scattering, the third term that of the third scattering and so on.If we consider a continuous distribution of scattering elements or so-called blobs, the sums have to be replaced by integrals as will be shown below for the case of troposcatter field representation. An inhomogeneous medium is given with a dielectric constant:

$$\varepsilon = \overline{\varepsilon}_0 + \hat{\varepsilon}(\vec{r},\xi)$$

$\overline{\varepsilon}_0$ is the mean value and $\hat{\varepsilon}(\vec{r},\xi)$ describes its random fluctuations, $\xi$ being a stochastic ensemble parameter. The vector $\vec{r}=(x,y,z)$ gives the position of $\hat{\varepsilon}$. The elimination of the magnetic field from MAXWELL's equations leads to a wave equation for the electric field vector.

$$\nabla^2\vec{E}(\vec{r},t) + \nabla(\tfrac{1}{\varepsilon}\vec{E}(\vec{r},t)\cdot\nabla\varepsilon) = \\ = \varepsilon\mu \frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2} \qquad (27)$$

If the changes of $\hat{\varepsilon}/\overline{\varepsilon}_0$ are small over one wave length of the high-frequency carrier signal, the gradient term may be neglected. Thus the coupling between the components of the electric field strength and the resulting depolarisation effects are not taken into considerantion. For each component of $\vec{E}$ the wave equation can be put into the following form:

$$(\nabla^2 - \varepsilon_0\mu_0 \frac{\partial^2}{\partial t^2})\, E(\vec{r},t) = \\ = \hat{\varepsilon}(\vec{r},\xi)\mu_0 \frac{\partial^2}{\partial t^2} E(\vec{r},t) \;. \qquad (28)$$

This inhomogeneous wave equation may be formally treated as if the right hand side were a known function. We get the formal "solution":

$$E(\vec{r},t) = E_i(\vec{r},t) + \\ + \mu_0 \int \varepsilon(\vec{r},\xi) G(\vec{r},\vec{r}\,',t,t')\frac{\partial^2}{\partial t^2} E(\vec{r}\,',t')\, d^3r'\, dt' \qquad (29)$$

with the GREEN's function:

$$G(\vec{r},\vec{r}\,',t,t') = -\frac{1}{4\pi} \frac{\delta(t'-(t-\frac{1}{c}|\vec{r}-\vec{r}\,'|))}{|\vec{r}-\vec{r}\,'|} \qquad (30)$$

The term $E_i(\vec{r},t)$ represents the primary incident field, that is the solution of the homogeneous wave equation in the absence of the random component $\hat{\varepsilon}(\vec{r},\xi)$.

For narrow-band signals, where the carrier frequency $f_c$ is large compared to the bandwidth of the modulation we may write:

$$\frac{\partial^2 E(\vec{r},t)}{\partial t^2} \approx - \omega_c^2 E(\vec{r},t) \qquad (31)$$

Equation (29) yields a scalar integral equation for each component of the unknown wave function $E(\vec{r},t)$. The NEUMANN-LIOUVILLE-expansion may be applied:

$$E(\vec{R},t) = E_i(\vec{R},t) -$$

$$- \omega_c^2 \mu_0 \int \hat{\varepsilon}(\vec{r},\xi) E_i(\vec{r},t') G(\vec{R},\vec{r},t,t') d^3r dt' +$$

$$+ (\omega_c^2 \mu_0)^2 \iint G(\vec{R},\vec{r},t,t') \hat{\varepsilon}(\vec{r},\xi) \cdot \qquad (32)$$

$$\cdot \; G(\vec{r},\vec{r}',t',t'') \hat{\varepsilon}(\vec{r}',\xi) E_i(\vec{r}',t'') d^3r' dt'' d^3r dt'$$

$$= E_i(\vec{R},t) + E^{(1)}(\vec{R},t) + E^{(2)}(\vec{R},t) t \ldots \quad .$$

The second term $E^{(1)}(\vec{R},t)$ is the so-called BORN approximation for the scattered field. The third term $E^{(2)}(\vec{R},t)$ describes the effect of two successive scatterings at two volume elements. An incident wave falling on the first scattering element $\hat{\varepsilon}(\vec{r}')$ is scattered and reradiated to $\vec{r}$ by $G(\vec{r},\vec{r}',t',t'')$, the second element $\hat{\varepsilon}(\vec{r})$ reradiates the result to the receiver via $G(\vec{R},\vec{r},t,t')$. The double integral sums the contributions over all possible pairs of scattering elements. Further terms represent the contribution of third order scatterings and so on. From (32) the space-time-correlation function of the scattered field $\langle E(\vec{R}_1,t_1) E(\vec{R}_2,t_2) \rangle$ may be calculated, using symbolic shorthand notations of stochastic system operators as applied by LIU [1], TATARSKI [5] and MIDDLETON [6].

## 8. CONCLUSION

A short summary has been given of the theory of linear spatial stochastic systems and associated random integral equations. The theory may help to describe and to understand wave propagation phenomena in a random medium.

It gives insight into the physical propagation mechanism, as has been shown in the present paper for the case of multiple scattering of time-dependent electromagnetic waves in a turbulent atmosphere.

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