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A New Family of Discrete Unitary Transforms and Their Potential Applications

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RESUME

On a developpe une nouvelle famille de transformations discretes et ortogonales (unitaires) en stipulant que la representation de la transformation reelle discrete Fourier soit formulee par une nouvelle groupe de fonctions basics formant 2 matrices factoriels etant orthogonal. La premiere transformation consiste des nouvelles fonctions basics comme une convolution circulaire. Les antres transformations santen analogie avec les transformations reelles discretes Fourier, sinus, cosinus symmetrique, et discretes Fourier en replacant $\sin 2\pi nk/N$ et $\cos 2\pi nk/N$ avec $z(nk/N)$ et $z(nk/N+1/4)$, respectivement, on $z(.)$, avec elements de 0 et puissance de $\sqrt{2}$, est la fonction construee pour obtenir des matrices orthonormales.
Avantages et applications possibles des transformations sant discutes.

SUMMARY

A new family of discrete orthogonal (unitary) transforms are developed by requiring that the representation of the real discrete Fourier transform in terms of a new set of basis functions results in 2 matrices of factorization which are orthonormal. The first transform consists of the new basis functions in the form of a circular convolution. The other transforms are similar to real discrete Fourier, sine, symmetric cosine, and discrete Fourier transforms with the replacement of $\sin 2\pi nk/N$ and $\cos 2\pi nk/N$ by $z(nk/N)$ and $z(nk/N+1/4)$, respectively, where $z(.)$, with elements of 0 and powers of $\sqrt{2}$, is the function constructed to achieve orthonormal matrices.
Advantages and potential applications of the transforms are discussed.

A New Family of Discrete Unitary Transforms and Their Potential Applications

1. INTRODUCTION

Discrete trigonometric transforms are very important in a variety of applications. In a previous article, we discussed real discrete Fourier transform (RDFT), which corresponds to Fourier series for sampled periodic signals with sampled, periodic frequency responses just as discrete Fourier transform (DFT) corresponds to complex Fourier series for the same type of signals [1]. RDFT is better than DFT in data compression, Wiener-filtering and computation of real convolution. RDFT is also very useful in designing fast transforms which approximate Karhunen-Loeve transform more optimally than other orthogonal transforms such as discrete cosine transform (DCT) [2].

RDFT can be represented in terms of Möbius basis functions instead of sines and cosines [3]. This results in the factorization of the RDFT matrix into 2 matrices. The first matrix, with elements 1, -1 and 0, is obtained by replacing $\cos 2\pi k/N$ and $\sin 2\pi k/N$ by $\mu(\frac{k}{n} + \frac{1}{4})$ and $\mu(\frac{k}{n})$, where $\mu(x)$ is the bipolar rectangular wave function. The second matrix is block-diagonal where each block is a circular correlation and consists of the Möbius basis functions.

DFT and DCT are related to RDFT by simple orthonormal matrices [4], [5]. RDFT is, in turn, equal to the direct sum of discrete symmetric cosine (DSCT) and sine (DST) transforms after preprocessing of the data vector with a simple orthonormal matrix [1]. As a consequence, it is possible to factorize all the discrete trigonometric transforms in the same way using Möbius basis functions.

It is desirable that the two matrices of factorization are also orthonormal, thereby defining new discrete orthogonal transforms which also factorize discrete trigonometric transforms. In order to achieve this, the theoretical development in [3] will be generalized by replacing the bipolar rectangular wave function $\mu(x)$ by another function $z(x)$ to be defined. $z(x)$ will be chosen to have the same properties as $\mu(x)$. The number of data points N will be restricted to powers of 2 in this introductory article. For such N, some of the properties of $z(.)$ (or $\mu(.)$) are [3]

$$z\left(\frac{n}{N}\right) = -z\left(\frac{n}{N} + \frac{1}{2}\right) \quad (1)$$

$$z\left(\frac{n}{N} + \frac{1}{4}\right) = z\left(\frac{N-n}{N} + \frac{1}{4}\right) \quad (2)$$

$$z\left(\frac{n}{N}\right) = -z\left(\frac{N-n}{N}\right) \quad (3)$$

With respect to these equations, $z(\frac{n}{N} + \frac{1}{4})$ and $z(\frac{n}{N})$ are isomorphic to $\cos 2\pi n/N$ and $\sin 2\pi n/N$, respectively.

$\sin 2\pi n/N$ and $\cos 2\pi n/N$ will be represented in terms of the new basis functions $b(m_1,N)$ as before, given by

$$\sin 2\pi\frac{n}{N} = \sum_{m_1=1}^{N-3} b(m_1,N)\, z\left(\frac{m_1 n}{N}\right) \quad (4)$$

$$\cos 2\pi\frac{n}{N} = \sum_{m_1=1}^{N-3} b(m_1,N)\, z\left(\frac{m_1 n}{N} + \frac{1}{4}\right) \quad (5)$$

where m_1 equals 1,5,9 --- (N-3).

It is seen that the composition of sines and cosines in terms of the new basis functions follows exactly the same procedure as before, with the replacement of $\mu(.)$ by $z(.)$. As a consequence, the subsequent development of the matrices of factorization also follows the same procedure, with the additional requirement that they be orthonormal.

The final results will be two types of transforms. The first one consists of the new basis functions in the form of a circular convolution. The second type of transforms will be analogous to RDFT, DSCT, DST and DFT with the replacement of $\sin 2\pi nk/N$ and $\cos 2\pi nk/N$ by $z(\frac{nk}{N})$ and $z(\frac{nk}{N} + \frac{1}{4})$, respectively. The values of $z(.)$ are 0, and powers of $\sqrt{2}$.

2. THE FUNCTION Z(.) AND THE NEW TRANSFORMS

The function $z(.)$ will be constructed to satisfy (1) thru (3). Setting n equal to 0 in (5) and requiring that the sum of the basis functions equals 1 as before [3] gives

$$z\left(\frac{1}{4}\right) = 1 \quad (6)$$

Since $z(.)$ is isomorphic to $\mu(.)$, it has to be periodic with period 1. Using (4) twice with the parameters $\frac{n}{N}$ and $\frac{2n}{2N}$ indicates that the following equation is still true:

$$b(m_1,N) + b\left(m_1 + \frac{N}{2},N\right) = b\left(m_1,\frac{N}{2}\right) \quad (7)$$

RDFT is given by [1]

$$\frac{x_1(x)}{v(n)} = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} x(k)\cos\frac{2\pi nk}{N} \quad (8)$$

$$x_0(m) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} x(k)\sin\frac{2\pi mk}{N} \quad (9)$$

where

$$n = 0,1,2 \text{ --- } N_1 \quad (10)$$

$$N_1 = \frac{N}{2} \quad \text{if N even} \quad (11)$$

$$= \frac{N-1}{2} \quad \text{if N odd}$$

$$m = 1,2,3, \text{ --- } N_0 \quad (12)$$

$$N_0 = \frac{N}{2} - 1 \quad \text{if N even} \quad (13)$$

$$= \frac{N-1}{2} \quad \text{if N odd}$$

$$v(n) = 1 \quad n \neq 0, \frac{N}{2} \quad (14)$$

$$= \frac{1}{\sqrt{2}} \quad n = 0, \frac{N}{2}$$

Using (4) and (5), (8) and (9) with N equal to a power of 2 can be written as

$$x_1(n) = \sum_{m_1=1}^{N-3} b(m_1,N)\, h_1(m_1 n \bmod N) \quad (15)$$

$$x_0(m) = \sum_{m_1=1}^{N-3} b(m_1,N)\, h_0(m_1 m \bmod N) \quad (16)$$

where

$$\frac{h_1(n)}{v(n)} = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} x(k)\, z\left(\frac{nk}{N} + \frac{1}{4}\right) \quad (17)$$

$$h_0(m) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} x(k)\, z\left(\frac{mk}{N}\right) \quad (18)$$

Assume that n and m in (15) thru (18) run between the limits 0 and N-1 with the understanding that $x_1(n) = x_1(N-n)$ and $x_0(m) = -x_0(N-m)$. Doing the permutations n or $m \to 2^m a^n \bmod N$, $m_1 \to a^k \bmod N$, where a equals -3 or N-3, and using (7), (15) and (16) can be written as [3]

$$\frac{x_1(2^m a^n \bmod N)}{v(2^m a^a \bmod N)} = \sqrt{\frac{2}{N}} \sum_{k=0}^{M_1-1} b(a^k,M)\, h_1(2^m a^{n+k} \bmod N) \quad (19)$$

$$x_0(2^m a^n \bmod N) = \sqrt{\frac{2}{N}} \sum_{k=0}^{M_1-1} b(a^k,M)\, h_0(2^m a^{n+k} \bmod N) \quad (20)$$

where

$$M = N/2^m \quad (21)$$

$$M_1 = M/4 \qquad M \geq 4 \tag{22}$$
$$\quad = 1 \qquad M < 4$$

$$m = 0,1,2,\ \text{---}\ \log_2 N \qquad \text{(equation (19))} \tag{23}$$
$$\quad = 0,1,2,\ \text{---}\ \log_2 N-2 \qquad \text{(equation (20))}$$

$$n = 0,1,2,\ \text{---}\ M_1-1 \tag{24}$$

(19) and (20) define a real matrix equation of size N where circular correlations of sizes given by different values of M_1 are computed. For example, when N = 16, equations (19) and (20) in matrix form are given by

$$\begin{bmatrix} \sqrt{2}a_1(0) \\ \sqrt{2}a_1(8) \\ a_1(4) \\ a_1(2) \\ a_1(10) \\ a_1(1) \\ a_1(13) \\ a_1(9) \\ a_1(5) \\ a_0(4) \\ a_0(2) \\ a_0(10) \\ a_0(1) \\ a_0(13) \\ a_0(9) \\ a_0(5) \end{bmatrix} = \frac{1}{\sqrt{8}} \left[\begin{array}{cccccccccccccccc} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&a(1,8)&a(5,8)&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&a(5,8)&a(1,8)&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&a(1,16)&a(13,16)&a(9,16)&a(5,16)&0&0&0&0&0&0&0 \\ 0&0&0&0&0&a(5,16)&a(1,16)&a(13,16)&a(9,16)&0&0&0&0&0&0&0 \\ 0&0&0&0&0&a(9,16)&a(5,16)&a(1,16)&a(13,16)&0&0&0&0&0&0&0 \\ 0&0&0&0&0&a(13,16)&a(9,16)&a(5,16)&a(1,16)&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&a(1,8)&a(5,8)&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&a(5,8)&a(1,8)&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&a(1,16)&a(13,16)&a(9,16)&a(5,16) \\ 0&0&0&0&0&0&0&0&0&0&0&0&a(5,16)&a(1,16)&a(13,16)&a(9,16) \\ 0&0&0&0&0&0&0&0&0&0&0&0&a(9,16)&a(5,16)&a(1,16)&a(13,16) \\ 0&0&0&0&0&0&0&0&0&0&0&0&a(13,16)&a(9,16)&a(5,16)&a(1,16) \end{array}\right] \begin{bmatrix} b_1(0) \\ b_1(8) \\ b_1(4) \\ b_1(2) \\ b_1(10) \\ b_1(1) \\ b_1(13) \\ b_1(9) \\ b_1(5) \\ b_0(4) \\ b_0(2) \\ b_0(10) \\ b_0(1) \\ b_0(13) \\ b_0(9) \\ b_0(5) \end{bmatrix} \tag{25}$$

Using (2) and (3), (17) and (18) can be written as

$$\frac{h_1(n)}{v(n)} = \frac{2}{\sqrt{N}} \sum_{k=0}^{N_1} g_1(k)v(k)z\left(\frac{nk}{N} + \frac{1}{4}\right) \tag{26}$$

$$h_0(m) = \frac{2}{\sqrt{N}} \sum_{k=1}^{N_0} g_0(k)z\left(\frac{nk}{N}\right) \tag{27}$$

where

$$g_1(0) = x(0) \tag{28}$$

$$g_1\left(\frac{N}{2}\right) = x\left(\frac{N}{2}\right) \tag{29}$$

$$g_1(k) = \frac{x(k) + x(N-k)}{\sqrt{2}},\ k \neq 0,\ \frac{N}{2} \tag{30}$$

$$g_0(k) = \frac{x(k) - x(N-k)}{\sqrt{2}} \tag{31}$$

Doing the same permutations as in the case of (15) and (16), (26) and (27) become [3]

$$\frac{h_1(2^m a^a \bmod N)}{v(2^m a^n \bmod N)} = \frac{2}{\sqrt{N}} \sum_{l=0}^{\log_2 N} \sum_{k=0}^{M_1-1} g_1(2^l a^k)v(2^l a^k)z\left(\frac{a^{n+k}}{M_2} + \frac{1}{4}\right) \tag{32}$$

$$h_0(2^m a^n \bmod N) = \frac{2}{\sqrt{N}} \sum_{l=0}^{\log_2 N} \sum_{k=0}^{M_1-1} g_0(2^l a^k)z\left(\frac{a^{n+k}}{M_2}\right) \tag{33}$$

where M_1, m and n are as defined in (22), (23) and (24), and

$$M_2 = \frac{N}{2^{m+1}} \qquad 2^m \leq \frac{N}{2^l} \tag{34}$$
$$\quad = 1 \qquad 2^m > \frac{N}{2^l}$$

z(.) should be designed such that (26) and (27) (or (32) and (33)) represent 2 orthonormal transforms. Then (17) and (18) together also represent an orthonormal transform since (28) thru (31) correspond to an orthonormal matrix preprocessing the input data vector. There are 2 more orthonormal transforms given by (19) and (20) since RDFT as a whole is orthonormal.

In order to see how to proceed further, let (32) and (33) be written with z(.) replaced by μ(.). For N=16, one gets

$$\begin{bmatrix} h_1(0) \\ h_1(8) \\ h_1(4) \\ h_1(2) \\ h_1(10) \\ h_1(1) \\ h_1(13) \\ h_1(9) \\ h_1(5) \\ h_0(4) \\ h_0(2) \\ h_0(10) \\ h_0(1) \\ h_0(13) \\ h_0(9) \\ h_0(5) \end{bmatrix} = \frac{1}{2} \left[\begin{array}{cccccccccccccccc} \frac{1}{2}&\frac{1}{2}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&0&0&0&0&0&0 \\ \frac{1}{2}&\frac{1}{2}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&0&0&0&0&0&0&0 \\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&1&-1&-1&0&0&0&0&0&0&0&0&0&0&0 \\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&-1&0&0&1&-1&1&-1&0&0&0&0&0&0&0 \\ \frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&-1&0&0&-1&1&-1&1&0&0&0&0&0&0&0 \\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&0&1&-1&1&1&-1&-1&0&0&0&0&0&0&0 \\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&0&-1&1&1&-1&-1&1&0&0&0&0&0&0&0 \\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&0&1&-1&-1&-1&1&1&0&0&0&0&0&0&0 \\ \frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}&0&-1&1&-1&1&1&-1&0&0&0&0&0&0& \\ 0&0&0&0&0&0&0&0&0&0&0&1&1&1&1& \\ 0&0&0&0&0&0&0&0&0&1&1&1&-1&1&-1& \\ 0&0&0&0&0&0&0&0&0&1&1&-1&1&-1&1& \\ 0&0&0&0&0&0&0&0&1&1&-1&1&-1&-1&1& \\ 0&0&0&0&0&0&0&0&1&-1&1&-1&-1&1&1& \\ 0&0&0&0&0&0&0&0&1&1&-1&-1&1&1&-1& \\ 0&0&0&0&0&0&0&0&1&-1&1&1&1&-1&-1& \end{array}\right] \begin{bmatrix} g_1(0) \\ g_1(8) \\ g_1(4) \\ g_1(2) \\ g_1(10) \\ g_1(1) \\ g_1(13) \\ g_1(9) \\ g_1(5) \\ g_0(4) \\ g_0(2) \\ g(10) \\ g_0(1) \\ g_0(13) \\ g_0(9) \\ g_0(5) \end{bmatrix} \tag{35}$$

The dotted lines indicate the circular correlations of sizes 2 and 4. Let the corresponding circular matrices of size M be denoted by A(M) for the real part and C(M) for the imaginary part. If A(2), C(2), A(4), C(4) in (35) are multiplied by $1/\sqrt{2}$, the resulting total matrix is orthonormal. Thus, they should be chosen to be

$$A(2) = C(2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{36}$$

$$A(4) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \quad (37) \qquad C(4) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \quad (38)$$

The way (35) is orthonormalized is not unique. The columns or rows of A(.) and C(.) can be permuted in any circular order without affecting the orthonormality of the total matrix. A(4) and C(4) can also be chosen as

$$A(4) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \quad (39) \qquad C(4) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \quad (40)$$

Circular permutations of rows or columns are allowed in this case as well. However, only the given orders will be considered.

The two possible ways of choosing A(M) and C(M) will be referred to as case I and case II. In either case, they are circular and of the form

$$D = \begin{bmatrix} E & -E \\ -E & E \end{bmatrix} \tag{41}$$

E being skew-circular.

For N > 16, it is not difficult to show that the total matrix as in (35) is orthonormal if A(M) and C(M) are generated from A(4) and C(4) by filling in zeroes between the original elements and multiplying the original elements by $\sqrt{M/8}$ in case I and $\sqrt{M/4}$ in case II such that the sum of the squares of the elements of each column or row equals $M/2$.

Let $A(M)$ and $C(M)$ be written as $(a_0 a_1 --- a_{M/4-1})_M$ and $(c_0 c_1 --- c_{M/4-1})_M$. The elements have the following values for the two cases:

$M = 1$: (both cases)

$$a_0 = c_0 = 1 \qquad (42)$$

$M = 2$: (both cases)

$$a_0 = -a_1 = c_0 = -c_1 = \frac{1}{\sqrt{2}} \qquad (43)$$

$M \geq 4$:

Case I

$$a_0 = a_{M/4} = -a_{M/2} = -a_{3M/4} = \sqrt{\frac{M}{8}} \qquad (44)$$

$$a_k = 0 \quad \text{otherwise} \qquad (45)$$

$$c_0 = -c_{M/4} = -c_{M/2} = c_{3M/4} = \sqrt{\frac{M}{8}} \qquad (46)$$

$$c_k = 0 \quad \text{otherwise} \qquad (47)$$

Case II

$$a_{M/4} = -a_{3M/4} = \frac{\sqrt{M}}{2} \qquad (48)$$

$$a_k = 0 \quad \text{otherwise} \qquad (49)$$

$$c_0 = -c_{M/2} = \frac{\sqrt{M}}{2} \qquad (50)$$

$$c_k = 0 \quad \text{otherwise} \qquad (51)$$

Comparing (32) and (33) with (35) thru (51) indicates that $z(a^k/4M)$ should assume the following values:

$M \leq 2$ (both cases)

$$z(0) = z(\tfrac{1}{2}) = 0 \qquad (52)$$

$$z(\tfrac{1}{4}) = 1 \qquad (53)$$

$$z(\tfrac{1}{8}) = z(\tfrac{3}{8}) = \frac{1}{\sqrt{2}} \qquad (54)$$

$M \geq 4$

Case I:

k	0	$M/4$	$M/2$	$3M/4$	otherwise
$z(a^k/4M)$	$\sqrt{M/8}$	$-\sqrt{M/8}$	$-\sqrt{M/8}$	$-\sqrt{M/8}$	0

(55)

Case II

k	0	$M/2$	otherwise
$z(a^k/4M)$	$\sqrt{M/2}$	$-\sqrt{M/2}$	0

(56)

In computer computations, multiplications with powers of 2 are much more simple than multiplications with powers of $\sqrt{2}$ since they are simply shift operations. Except for $A(2)$ and $C(2)$, $A(M)$ and $C(M)$ can be chosen to contain powers of 2 only by choosing case I when $8 \mid M$ and case II when $8 \nmid M$. This possibility can be called case III.

3. THE NEW BASIS FUNCTIONS

Since $z(2^l a^k/N)$ equals 0 for most k, as seen in (57)and (58), finding the new basis functions $b(m,N)$ becomes a relatively easy task. However, the three cases have to be considered separately. Case II is discussed below.

There are groups of basis functions related through equations according to $b(a^k,N)$, $b(a^k + \frac{N}{2},N)$ for $k = 0,1,---(N/8-1)$. For each k, (7) yields

$$b(a^k,N) + b(a^k + \tfrac{N}{2},N) = b(a^k,\tfrac{N}{2}) \qquad (57)$$

Using (4) and (56), one obtains

$$\sin\frac{2\pi}{N} a^l = \frac{\sqrt{N}}{4}\left[b(a^k,N) - b(a^k + \tfrac{N}{2},N)\right] \qquad (58)$$

where l is defined by

$$l = (N/4-k) \bmod N/4 \qquad (59)$$

The solution of (57) and (58) becomes

$$\begin{bmatrix} b(a^k,N) \\ b(a^k + \frac{N}{2},N) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} b(a^k,\frac{N}{2}) \\ \frac{4}{\sqrt{N}}\sin\frac{2\pi}{N}a^l \end{bmatrix} \qquad (60)$$

which is recursive. The values of $b(1,4)$, $b(1,8)$ and $b(5,8)$ can be shown to be 1,1 and 0, respectively, by comparing (8) and (9) to (19) and (20).

4. DISCUSSION

The discrete orthogonal transforms which result from the development in the previous sections will be discussed below under the code names T_1, T_2, T_3, T_4 and T_5.

T_1

(15) defines an orthogonal transformation with a transformation matrix which is circular.

The following correspondence will be defined:

$$b(a^k,L) \leftrightarrow b((\tfrac{L}{4} - k) \bmod \tfrac{L}{4}, \tfrac{L}{4})$$

Then T_1 can be written as

$$y(n) = \sum_{k=0}^{N-1} x(k)\, b((n-k) \bmod N, N) \qquad (61)$$

(74) is a circular convolution. It can be written in matrix form as

$$y = Bx \qquad (62)$$

where B is left-circular. As a consequence, it is similar to the DFT matrix F:

$$B = F^* \Lambda F \qquad (63)$$

where Λ is the diagonal matrix of the eigenvalues of B [7]. Since B is orthonormal, it follows that

$$BB^T = \Lambda\Lambda^* = I \qquad (64)$$

I being the identity matrix. According to the last equation, the eigenvalues are roots of unity. The eigenvectors are the columns of F. B can be made right-circular according to

$$B' = PB \qquad (65)$$

where P is the permutation matrix corresponding to the interchange of the rows of B according to $n \leftrightarrow N-n$, $n=1,2$ --- $(N-1)$, n being the row index starting with 0. B' is symmetric in addition to being orthonormal. Hence, its eigenvectors are real and the eigenvalues are restricted to $\pm$ 1.

B can be written as the circulant $(b_0, b_1, \cdots b_{N-1})$. Using (7) and $b(1,4) = 1$, it is straightforward to show that

$$\sum_{k=0}^{N-1} b_k = 1 \quad (66)$$

Since B is orthonormal, it is also true that

$$\sum_{k=0}^{N-1} b_k^2 = 1 \quad (67)$$

T_2

(17) and (18) make up the second transform. These equations are analogous to (8) and (9), which define RDFT, with $z(\frac{nk}{N} + \frac{1}{4})$ and $z(\frac{nk}{N})$ replacing $\cos\frac{2\pi}{N}nk$ and $\sin\frac{2\pi}{N}nk$, respectively. The inverse transform can be similarly shown to be

$$x(k) = \sqrt{\frac{2}{N}}\left[\sum_{k=0}^{N_1} h_1(k)v(k)z(\frac{nk}{N} + \frac{1}{4}) + \sum_{k=1}^{N_0} x_0(k)z(\frac{nk}{N})\right] \quad (68)$$

T_3

When $x(k) = x(N-k)$, $h_0(.)$ is zero, and $h_1(.)$ is given by (26). This equation is orthonormal and thus defines the following transform:

$$\frac{y(n)}{v(n)} = \frac{2}{\sqrt{N}}\sum_{k=0}^{N_1} x(k)v(k)z(\frac{nk}{N} + \frac{1}{4}) \quad (69)$$

This equation is analogous to the equation which defines DSCT, with $z(\frac{nk}{N} + \frac{1}{4})$ replacing $\cos\frac{2\pi nk}{N}$.

The inverse transform is given by

$$x(n) = \sqrt{\frac{2}{N}}\sum_{k=0}^{N_1} y(k)v(k)z(\frac{nk}{N} + \frac{1}{4}) \quad (70)$$

T_4

When $x(k) = -x(N-k)$, $h_1(.)$ is zero, and $h_0(.)$ is given by (27), which is orthonormal, defining the following transform:

$$y(n) = \frac{2}{\sqrt{N}}\sum_{k=1}^{N_0} x(k)z(\frac{nk}{N}) \quad (71)$$

This equation is analogous to the equation which defines DST, with $z(\frac{nk}{N})$ replacing $\sin\frac{2\pi nk}{N}$.

The inverse transform is given by

$$x(n) = \frac{2}{\sqrt{N}}\sum_{k=1}^{N_0} y(k)z(\frac{nk}{N}) \quad (72)$$

T_5

Orthonormalized DFT is given by [1]

$$y(n) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x(k)\, w^{-nk} \quad (73)$$

where w equals $e^{2\pi j/N}$. Let the transformation matrices of DFT and RDFT be W and R, respectively. The two are related by [1]

$$W = W_1 R \quad (74)$$

where the elements of the matrix W_1 are given by

$$(W_1)_{0,0} = 1 \quad (75)$$

$$(W_1)_{\frac{N}{2},\frac{N}{2}} = 1 \qquad \text{if } N \text{ even} \quad (76)$$

$$(W_1)_{n,n} = \frac{1}{\sqrt{2}} = (W_1)_{N-n,n} \qquad 1 \le n \le N_0 \quad (77)$$

$$(W_1)_{n,N_1+n} = -\frac{j}{\sqrt{2}} = (W_1)_{N-n,N_1+n} \qquad 1 \le n \le N_0 \quad (78)$$

$$(W_1)_{m,n} = 0 \qquad \text{otherwise} \quad (79)$$

Let the transformation matrix of T_2 be R'. In analogy to DFT, it is possible to define

$$W' = W_1 R' \quad (80)$$

resulting in the transform given by

$$h(n) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} x(k)z_c(-nk) \quad (81)$$

with the inverse transform

$$x(n) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} h(k)z_c(nk) \quad (82)$$

where

$$z_c(\pm m) = z(\frac{m}{N} + \frac{1}{4}) \pm jz(\frac{m}{N}) \quad (83)$$

It is seen that T_5 is similar to DFT with w replaced by $z_c(.)$. The relationship between R and R' can be written as

$$R = B_T R' \quad (84)$$

where B_T is the matrix corresponding to the direct sum of circular correlations as defined by (19) and (20). It is not difficult to show that W_1 and B_T commute. As a consequence, becomes

$$W = B_T W_1 R' \quad (85)$$

, which defines DFT, can now be written as

$$y(n) = \sum_{m_1=1}^{N-3} b(m_1,N)\, h(m_1 n \bmod N) \quad (86)$$

This equation shows the relationship between DFT, T1 and T5.

Since there are 3 possible ways to choose z(.) and b(.,N), the corresponding transforms T1 thru T5 can be considered to be of the first, second and third kind, respectively.

The matrix elements of the transforms T2 thru T5 are simple containing 0 and powers of $\sqrt{2}$ (odd powers suppressed in the third kind). In this regard, they are similar to discrete orthogonal transforms with simple elements [6]. Especially discrete Haar transform (DHT) has elements 0 and powers of $\sqrt{2}$. However, there are notable differences. The discrete orthogonal transforms like DHT are defined by an equation of the form

$$y(n) = \sum x(k)f(n, \frac{k}{N}) \quad (87)$$

which means that for each n, there is a different waveform to be sampled. This can be compared to the equations defining T2 thru T5 where only a single waveform z(.) is sampled as in the case of discrete trigonometric transforms.

As a comparison, the matrices corresponding to T2 and DHT for N=8 are shown below:

T_2

$$\frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 1 & 0 & -1 & -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 & 1 & -1 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 & -1 \end{bmatrix}$$

DHT

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

5. COMPUTATIONAL CONSIDERATIONS

Computation of T_1 involves circular convolution. There is a considerable number of techniques developed for computation of circular convolutions in recent years [8].

Since T_2 is analogous to RDFT in structure, it can be computed in terms of skew-circular correlations (SCC's) [9]. Since the numbers of nonzero elements ($M \geq 4$) are 4 in case I and 2 in case II, each SCC of size M/2 actually reduces to M/4 independent SCC's of sizes 2 and 1 in the two cases, respectively. The computation of the other transforms can be done similarly. Thus, all 5 transforms are in the class of fast transforms.

An immediate implication of the above procedure for the computation of T_2 is the simplification of the computation of RDFT. (25) shows an example of circular correlations of various sizes after T_2 computation. In practise, the size of the largest circular correlation which can be computed by available means may be limited, say, to N_c. The larger correlations can be reduced to size N_c by repeated use of T_2 and the theorem for circular convolution using RDFT or DFT [1].

6. POTENTIAL APPLICATIONS

Fast transforms are useful in diverse technological problems. Thus, the transforms decribed above may find applications in a number of areas such as fast computation of DFT and RDFT, data compression, digital filtering, computation of convolution and deconvolution, optical computing and signal processing, VLSI implementation of transforms, pattern recognition, image processing, robot vision, sonar and radar, parallel architectures, mathematical and statistical methods, seismic data processing, chemistry and spectroscopy, medicine, and control systems.

One important advantage of factorizing RDFT and DFT in the form described above can be that transforms of type T2 can first be used for coarse processing of data, for example, in signal detection or classification. Further processing in the form of Fourier spectral components can be done, only if need be, by use of T1.

7. CONCLUSIONS

In this article, the representation of RDFT in terms of new basis functions leading to 2 matrices of factorization is further generalized with the requirement that the matrices be orthonormal. This procedure resulted in new discrete orthogonal transforms with interesting properties. The first transform, T_1, is a circular convolution. The other transforms, T_2, T_3, T_4 and T_5 are analogous to RDTF, DSCT, DST and DFT with the replacement of $\sin\frac{2\pi nk}{N}$ and $\cos\frac{2\pi nk}{N}$ by $z(\frac{nk}{N})$ and $z(\frac{nk}{N} + \frac{1}{4})$, respectively.

Another way to interpret the results is that, if the input data is in terms of the basis vectors given by the columns of the T_2 matrix, and the output frequency components are in terms of the standard basis vectors, RDFT reduces to the direct sum of T_1's.

Since $z(.)$ has simple values, T_2 thru T_5 can be expected to be useful in a number of applications such as real-time signal and image processing, pattern recognition and artificial intelligence. One advantage they have in comparison to similar transforms such as DHT is that the results can be converted to Fourier spectral information, if desired, by further processing with direct sum of T_1's, which are of sizes N/4, N/8 ----- 2, N being the size of the initial vector space.

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