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Precise Connection Between Ladder and Kalman Type
Adaptive Estimation Algorithms

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RESUME

On analyse les algorithmes de carrés minimes Kalman et "ladder", pour trouver la relation exacte entre les deux. On voit qu'ils ne sont pas équivalents en général. Les algorithmes "ladder" imposent une fenetre sur la signal.

SUMMARY

An analysis of the Kalman type and ladder least squares algorithms is performed towards the establishment of the precise connection between them. It is seen that they are not equivalent in a general sense. Windowing of the data is imposed as a condition for optimality of the ladder algorithms.



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Introduction:

In the past few years several ladder algorithms have been presented for solving least squares problems [1-3]. These ladder structures have been studied in different contexts, specially as an alternative realization of a digital filter on one side, and as a natural result in a least squares optimization problem on the other.

In this paper we explore the exact relation between the least squares ladder algorithm, and Kalman type least squares algorithms, for the case in which both seem to solve the same problem, viz. recursive estimation of the AR model parameters for a given string of data by minimization of the sum of squares of the output error sequence.

The connection has been loosely mentioned in the literature but not clearly established. Sometimes ladder and AR forms are shown to be equivalent whitening filters obtained by minimization of the cost functional defined by the sum of squared errors for the same set of data. But nothing is said about the connection in the case of the update of the recursive implementations. In some other cases both recursive algorithms are treated as equivalent in the sense of optimality but the relation is not established. And still in other works they are said to be almost identical but the "probable" difference is not explained [1-6].

In this presentation we analyse the AR scalar case of orders 1 and 2. The generalization to order n is the subject of future work.

AR model of order 1

The least squares solution for this case is

$$Q_T = - \left(\sum_{i=1}^T Z^2(i-1) \right)^{-1} \sum_{i=1}^T Z(i-1)Z(i) \quad (1)$$

Comparison of this expression with the ladder algorithm leads to the identification of

$$\sum_{i=1}^T Z^2(i-1) = R_0(T-1) \quad (2)$$

and

$$\sum_{i=1}^T Z(i-1)Z(i) = \Delta_1(T)$$

So that

$$Q_T = -R_0^{-1}(T-1)\Delta_1(T) = -K_1^r(T) \quad (3)$$

and both algorithms are equivalent at each step.

AR model of order 2

The least squares solution in this case is

$$Q_T = - \left(\sum_{i=2}^T Z_i Z_i^t \right)^{-1} \sum_{i=2}^T Z_i Z(i) \quad (4)$$

where $Z_i = [Z(i-1), Z(i-2)]^t$ is the delayed data vector and $Z(i)$ the data points. Eq. (4) can be rewritten:

$$Q_T = -\frac{1}{\Gamma(T)} \begin{bmatrix} E(T) & -C(T) \\ -C(T) & A(T) \end{bmatrix} \begin{bmatrix} D(T) \\ B(T) \end{bmatrix} \quad (5)$$

where

$$\begin{aligned} A(T) &= \sum_{i=2}^T Z^2(i-1) \\ B(T) &= \sum_{i=2}^T Z(i)Z(i-2) \\ C(T) &= \sum_{i=2}^T Z(i-1)Z(i-2) \\ D(T) &= \sum_{i=2}^T Z(i)Z(i-1) \\ E(T) &= \sum_{i=2}^T Z^2(i-2) \\ \Gamma(T) &= A(T)E(T) - C^2(T) \end{aligned} \quad (6)$$

so that

$$Q_1^{(2)}(T) = \frac{C(T)B(T) - E(T)D(T)}{\Gamma(T)} \quad (7)$$

$$Q_2^{(2)}(T) = \frac{C(T)D(T) - A(T)B(T)}{\Gamma(T)}$$

The expression for the Kalman gains as a function of $A(T)$ to $E(T)$ is easily obtained by identifying the factor of $Z(T)$ in both eqs. (7) and gives

$$K_1(T) = \frac{C(T-1)Z(T-2) - E(T-1)Z(T-1)}{A(T)E(T) - C^2(T)} \quad (8)$$

$$K_2(T) = \frac{C(T-1)Z(T-1) - A(T-1)Z(T-2)}{A(T)E(T) - C^2(T)}$$

Using the equivalence expression between AR and ladder coefficients

$$Q_i^{(2)}(T) = - (K_1^r(T) - K_1^e(T-1)K_2^r(T)) \quad (9a)$$



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$$\alpha_2^{(2)}(\tau) = -K_2^r(\tau) \quad (9b)$$

we can expand them using the ladder algorithm equations and isolate the factors of $\mathbf{z}(\tau)$ in each. These ought to be the Kalman gains if both algorithms are exactly equivalent.

Starting from the definition

$$K_2^r(\tau) = \frac{\Delta_2(\tau)}{R_1^r(\tau)} \quad (10)$$

and substituting

$$\Delta_2(\tau) = \Delta_2(\tau-1) + \frac{\varepsilon_1(\tau) r_1(\tau-1)}{\delta_1(\tau)} \quad (11)$$

where $\delta_1(\tau) = \cos^2 \theta_{0,1,\tau}$ and

$$\varepsilon_1(\tau) = \mathbf{z}(\tau) - K_1^r(\tau) \mathbf{z}(\tau-1) \quad (12)$$

$$K_1^r(\tau) = \frac{\Delta_1(\tau)}{R_0^r(\tau-1)}$$

leads to the expression

$$K_2^r(\tau) = \frac{\Delta_2(\tau-1)}{R_1^r(\tau-1)} + \mathbf{z}(\tau) K_2'(\tau) - \frac{\Delta_1(\tau-1) r_1(\tau-1) \mathbf{z}(\tau-1)}{\delta_1(\tau) R_1^r(\tau-1) R_0^r(\tau-1)} \quad (13)$$

where

$$K_2'(\tau) = -\frac{R_0^r(\tau-2) r_1(\tau-1)}{\delta_1(\tau) R_0^r(\tau-1) R_1^r(\tau-1)} = -\frac{r_1(\tau-1)}{R_1^r(\tau-1)} \quad (14)$$

is the hypothetical Kalman gain.

The other terms of equation (13) must be shown to correspond to the ones of

$$\alpha_2^{(2)}(\tau) = \alpha_2^{(2)}(\tau-1) (1 + K_2(\tau) \mathbf{z}(\tau-2)) + \mathbf{z}(\tau) K_2(\tau) + K_2(\tau) \mathbf{z}(\tau-1) \alpha_1^{(2)}(\tau-1) \quad (15)$$

Using the fact that

$$\frac{R_1^r(\tau-2)}{R_1^r(\tau-1)} = 1 - \frac{r_1(\tau-1) \mathbf{z}(\tau-2)}{R_1^r(\tau-1) \delta_0(\tau-1)} + \frac{r_1(\tau-1) K_1^r(\tau-1) \mathbf{z}(\tau-1)}{R_1^r(\tau-1) \delta_0(\tau-1)} \quad (16)$$

where $\delta_0(\tau-1) = \cos^2 \theta_{0,0,\tau-1}$ and $\delta_0(\tau-1) = \delta_1(\tau)$, and the AR-ladder structures equivalence (9), those terms of (15) are found.

Expanding the second equation in (9) in an analogous way we get

$$K_1'(\tau) = -\left(\frac{\mathbf{z}(\tau-1)}{R_0^r(\tau-1)} + K_1^r(\tau-1) K_2'(\tau) \right) \quad (17)$$

For comparing gains of eq. (8) with (14) and (17), the initialization procedure must be taken into account.

The usual initialization is

$$\Delta_1(0) = 0 \quad (18)$$

$$\delta_0(\tau) = 1$$

$$R_0^E(0) = R_0^r(0) = \mathbf{z}^2(0)$$

so (15) with some algebra and use of ladder algorithm eqs. turns into

$$K_2'(\tau) = -\frac{R_0^E(\tau-1) \mathbf{z}(\tau-2) - \Delta_1(\tau-1) \mathbf{z}(\tau-1)}{R_0^r(\tau-2) R_0^E(\tau-1) - \Delta_1^2(\tau-1)} = \frac{C(\tau-1) \mathbf{z}(\tau-1) - A(\tau-1) \mathbf{z}(\tau-2) - \mathbf{z}^2(0) \mathbf{z}(\tau-2)}{E(\tau) A(\tau) - C^2(\tau) + \mathbf{z}^2(0) E(\tau)} \quad (19)$$

and (17) into

$$K_1'(\tau) = \frac{\Delta_1(\tau-1) \mathbf{z}(\tau-2) - R_0^r(\tau-2) \mathbf{z}(\tau-1)}{R_0^r(\tau-2) R_0^E(\tau-1) - \Delta_1^2(\tau-1)} = \frac{C(\tau-1) \mathbf{z}(\tau-2) - E(\tau-1) \mathbf{z}(\tau-1)}{A(\tau) E(\tau) - C^2(\tau) + \mathbf{z}^2(0) E(\tau)} \quad (20)$$

which are different from (8).

If the summation index of eqs. (6) is taken from $i=0$ to T instead of $i=2$, and $\mathbf{z}(-1)$, $\mathbf{z}(-2)$ are assumed zero then we have

$$R_0^E(\tau-1) = R_0^r(\tau-1) = E(\tau+1) = A(\tau) \quad (21)$$

and the gains for both algorithms are the same.

Conclusions:

We have analyzed the precise connection between the Kalman and ladder type least squares algorithms for the scalar AR model of orders 1 and 2.

We have shown that they are equivalent for order 1 but not always for order 2. In the latter case equivalence requires the first data points of the set to be zero. This is essentially the meaning of taking the sums from $i=0$ and the points $\mathbf{z}(-i)=0$. The optimal solution for the least squares problem for a given set of data is eq. (4). So the ladder algorithm is in general not step by step optimal unless the previous conditions are met. This surely accounts for the name "prewindowed ladder form" in [2] and other papers. But our analysis shows that the prewindowing is imposed on the data



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by the algorithm and is not an additional feature to be selected.

It's the statistical interpretation which are affected mostly by this fact, because step by step optimality would only be true for processes that give zero data points in the right places. We analyze in another paper the step by step maximum likelihood estimator for a data set that comes from an infinite length process

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