A MODIFIED ZOOM-TRANSFORM FOR HIGH-RESOLUTION FOURIER ANALYSIS

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RÉSUMÉ

Dans ce papier, une méthode est presentée pour obtenir un spectre de haute resolution a partir de transformations de Fourier a court terme d'un signal echantilloné. La methode est basée sur ladite Transformée ZOOM, qui doit etre modifié dans deux points importants afin d'obtenir des bons resultats. Nous donnons une description de la theorie, qui est expliquée et illuminée par des résultats de simulations.

ABSTRACT

A method is presented for generating a high resolution spectrum from blocks of short-time discrete Fourier transforms of a sampled time signal. It is based on the Zoom transform, which has however to be modified in two important points to obtain useful results. This paper gives a description of the theory, explains it and shows simulation examples.

1 Introduction

When telecommunication or measurement systems are being designed, the following question is often encountered:

• How can a spectral band with high resolution be obtained from a gapless or overlapping series of low resolution short-time spectra?

This spectral magnifier function, also called "Zoom" (see [1]), is of great interest for measurement technology and for the processing of sonar and radar signals.

To solve this problem, there are various possible methods, see [1,5]. Of these methods, the Zoom transform is investigated and described here in a generalised form. The classical Zoom transform as portrayed in [5] means that the complex short-time spectra are written one under another row-by-row, and a Fourier transform is then performed vertically over the same frequency cell of all spectra. The result - to express it in simplified terms -is then a spectral analysis over precisely that frequency band which corresponds to the frequency cell from the low resolution short-time spectrum. The fact that the real behaviour in this method is more complicated will become clear in this paper. Both processes together are a 2-dimensional DFT of a 1-dimensional signal. (In practical applications, one will naturally always try to implement an FFT.) Such a transform is computationally attractive and very useful for many applications in signal processing and measurement, especially for systems based on short-time Fourier transforms. However, the straightforward approach of using non-overlapping, non-windowed data blocks gives very bad results, see also [5] . This can be observed in Fig. 1 for the example of 16 short-time DFT's of 16 points of a complex sinusoid with frequency 0.975 Hz and a (normalized) sampling frequency of 1 Hz. Strong aliasing effects are visible. Windowing alone does not help. The intuitive reason is that a rectangularly windowed data block of length T has a bandwidth 2/T, and must therefore be sampled at least with rate T/2, which means 50 % overlap. In this paper, it is shown that by using 75 % overlapping and windowed DFT's, very good results are obtained. See Fig. 3 for the same DFT length as in Fig. 1. We call this method a **modified Zoom-Transform**. At this point, let us not miss the opportunity to refer to a related subject.

A relationship - though not a directly obvious one - exists with the method of the synthetic aperture [3, 4] from radar and sonar technology: by spatial movement of a short array (aperture), a long array (synthetic aperture) is formed. Now, as is well known, the beam pattern is the Fourier transform of the array weighting. If one identifies the successive short-time spectra in the case of the Zoom transform with the beam patterns of spatially shifted short arrays, the beam pattern of a long array is generated by the second Fourier transform (corresponding to a high resolution spectral representation). It is therefore no wonder that, in [3, 4], similar considerations concerning overlapping (of the shifted arrays) occur as in Chapter 3. However, the reference to the problem of read-out of the "correct" data is missing in [3, 4].

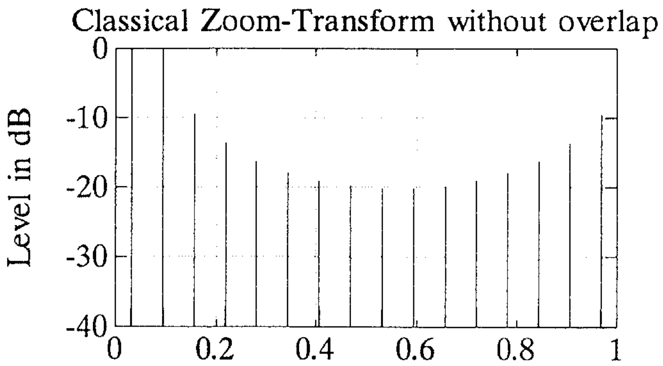

Figure 1; Frequency in Hz

2 Continuous and Discrete Fourier Transform

The starting point is a signal $s(t)$ which is sampled in the time window $0 \leq t < \tau$ in accordance with its bandwidth. Let the sampling interval be

$$t_s = \frac{\tau}{N} \tag{1}$$

Altogether, there are N sampling values $s(kt_s)$ where $0 \leq k \leq N-1$.

Firstly, let the discrete Fourier transform (DFT) of $s(t)$ in the time window $0 \leq t < \tau$, given by

$$DFT(s)(n) = \sum_{k=0}^{N-1} s(kt_s)e^{-i\frac{2\pi kn}{N}} \tag{2}$$

be related to the continuous Fourier transform $s\hat{}(f)$ of s(t), defined by

$$s\hat{}(f) = \int_{-\infty}^{\infty} s(t)e^{-i2\pi ft}dt \tag{3}$$

These considerations are not new, but they are an aid to understanding the two-dimensional considerations which follow later.

It is well known that, with the Woodward symbol rep (see [2])

$$rep_{t_s}g(t) = \sum_{j=-\infty}^{j=+\infty} g(t - jt_s) \tag{4}$$

the Fourier transform of a signal sampled in the lattice with the lattice point spacing t_s

$$g(t)rep_{t_s}\delta(t) \tag{5}$$

is

$$f_s rep_{f_s} g\hat{}(f), \tag{6}$$

i.e. it is the periodic repetition of the Fourier transform of g in the reciprocal lattice with the lattice point spacing $f_s = 1/t_s$.

The DFT of s in the time window $0 \leq t < \tau$ is then the Fourier transform of the time-windowed signal

$$s(t)rect(\frac{1}{\tau}(t - \tau/2)) \tag{7}$$

taken at the frequency points f_n, where

$$f_n = \frac{f_s}{N}n, \; 0 \leq n \leq N-1 \tag{8}$$

sampled at intervals t_s, i.e. altogether,

$$DFT(s)(n) = [\{s(t)rect(\frac{1}{\tau}(t - \tau/2))\}rep_{t_s}\delta(t)]\hat{}(\frac{f_s}{N}n) \tag{9}$$

3 Two-Dimensional Consideration of the Modified Zoom Transform

First of all, the terminology for overlapping short-time DFT's will now be introduced:

M: Number of short-time windows in the long-time window without overlapping

t_0: Overlap time of two consecutive windows

O: Degree of overlapping, $O = \frac{t_0}{\tau}$

m: Overlap parameter $m = \frac{1}{1-O}$, i. e.
$m = 1$ no overlapping, $m = \infty$ full overlap, $m = 2$ means 50 % overlap,
The overlap expressed as a percentage is $100(1 - 1/m)O$

Thus, in the case of overlapping, the position of the q-th short-time window is $(q-1)\frac{T}{Mm}$. There remains $\tau = T/M$, the length of the short-time analysis-window with N points. To describe the relationship between the discrete 2D-DFT of overlapping, weighted pieces of data, i.e. of the modified Zoom transform, and of a long DFT, the following auxiliary construction is introduced: By means of the transform

$$g(t_x, t_y) := s(t_x + t_y) \tag{10}$$

$s(t)$ is converted into a two-dimensional signal. The following applies:

$$\begin{aligned} s(t_x) &= g(t_x, 0) = s(t), \\ s(t_y) &= g(0, t_y) = s(t). \end{aligned} \tag{11}$$

Obviously, the rectangle

$$R(t_x,t_y) = rect(\frac{t_x}{\tau} - \frac{1}{2})rect(\frac{t_y}{T} - \frac{1}{2}) \tag{12}$$

contains all shifted data pieces of s, each of length τ. It can be seen that a 2D-DFT of the windows of length τ, shifted by $(q-1)\frac{\tau}{m}$, can be described as a 2D Fourier transform of

$$g(t_x,t_y)rect(\frac{t_x}{\tau} - \frac{1}{2})rect(\frac{t_y}{T} - \frac{1}{2}) \tag{13}$$

sampled at the lattice points at intervals of $\triangle t_x = t_s$ in the t_x direction and $\triangle t_y = \frac{\tau}{m}$ in the t_y direction. Because the sampling in the t_x direction is comparatively fine, whereas the sampling in the t_y direction is coarse, in the following we will call the t_x-axis t_{fi} and the t_y-axis t_{co}. The frequency values in the spectrum are then sampled at the points $(f_{co,l}, f_{fi,r})$, where

$$f_{co,l} = \frac{f_s}{N} l = f_\tau l,\ 0 \le l \le N-1 \tag{14}$$

and

$$f_{fi,r} = \frac{m/\tau}{Mm} r = f_T r,\ 0 \le r \le Mm-1\ . \tag{15}$$

Here f_τ is the frequency resolution of the window of length τ, and $f_T$is the frequency resolution of the window of length T, i.e.

$$f_\tau = \frac{1}{\tau} = \frac{f_s}{N},\ f_T = \frac{1}{T} = \frac{1}{\tau M} = \frac{f_\tau}{M} = \frac{f_s}{NM}.$$

Therefore, the following expression is obtained:

$$\begin{aligned}&\{g(t_{fi},t_{co})rect(\tfrac{t_{fi}}{\tau} - \tfrac{1}{2})rect(\tfrac{t_{co}}{T} - \tfrac{1}{2})\\ &\times rep_{t_s,\tau/m}\delta(t_{fi},t_{co})\}^{\hat{}\hat{}}(f_{co,l}, f_{fi,r}).\end{aligned} \tag{16}$$

From this, according to the calculation rules for sampling and Fourier transformation of two-dimensional signals, the following expression is obtained for the 2D-DFT:

$$\begin{aligned}s_{2D}{}^{\hat{}\hat{}}(l,r) &:= f_s m f_\tau \tau T rep_{f_s, mf_\tau}\{g^{\hat{}\hat{}}(f_{co},f_{fi})\\ &*sinc(\tfrac{f_{co}}{f_\tau})e^{-i2\pi\frac{f_{co}}{2f_\tau}} * sinc(\tfrac{f_{fi}}{f_T})e^{-i2\pi\frac{f_{fi}}{2f_T}}\}(f_{co,l}, f_{fi,r})\\ &= f_s m T rep_{f_s, mf_\tau}\{s^{\hat{}}(f_{fi})\delta(f_{fi}-f_{co})\\ &*sinc(\tfrac{f_{co}}{f_\tau})e^{-i2\pi\frac{f_{co}}{2f_\tau}} * sinc(\tfrac{f_{fi}}{f_T})e^{-i2\pi\frac{f_{fi}}{2f_T}}\}(f_{co,l}, f_{fi,r}).\end{aligned} \tag{17}$$

As a result of the continuous Fourier transform, $g(t_{fi},t_{co})$ is mapped onto $s^{\hat{}}(f_{fi})\delta(f_{fi}-f_{co})$, i.e. the values of $g^{\hat{}}(f_{fi},f_{co})$ exist only on the straight line $f_{fi} = f_{co}$, where they correspond to those of $s^{\hat{}}(f_{co})$.

Because the continuous Fourier transform does not make any distinction between coarse and fine frequency axis, $s(f_{fi})$ means the desired spectrum of $s(t)$, which then occurs again on the bisector of the angle between the two axes. Due to the windowing in the time domain, these values are blurred in the f_{co} and f_{fi} directions by convolutions with sinc functions, and because of the sampling they are repeated periodically at intervals of f_s and mf_τ, respectively. This two-dimensional signal is then sampled in the spectrum too - see Fig. 2. for $s(t) := e^{i2\pi 2.5 f_\tau t}$. A first consideration of the result indicates the following problems of the classical and modified Zoom transforms:

- The sampling points in the spectrum are generally not situated on the periodically repeated straight line $f_{co} = f_{fi}$.
- The number of points in the case of overlapping is m times higher than in the case of the simple DFT over the entire length T.

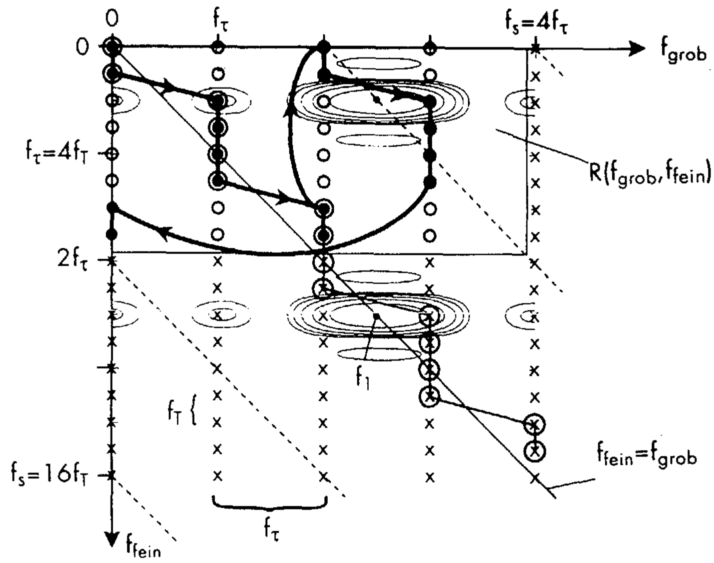

Figure 2

The question is now which NM points should be selected. Because the spectrum on the straight line cannot be sampled directly, it is then obvious that sampling points as close as possible to the straight line should be selected. However (with m = 2), only the first half of the staircase is situated in the original region $R(f_{co}, f_{fi})$ of the 2D-DFT; the second half is situated in the periodic repetitions. The points corresponding to this must then be selected from the original region. This produces the rather tortuous path indicated by a thick line in Fig. 2, which represents the best possible approximation to the desired spectrum. Formally, the following selection rule is applicable:

$$\begin{aligned}s_{Zoom}{}^{\hat{}}(n) = s_{2D}{}^{\hat{}\hat{}}(&\text{rem}\{\text{round}(n/M)\}\text{modulo } N),\\ &\text{rem}(n) \text{ modulo } mM)\\ &0 \le n \le NM-1.\end{aligned} \tag{18}$$

Another question is the following: What does the blurring function look like? To answer this, (17) is evaluated and the following slightly monstrous result is produced:

$$s_{2D}^{\wedge\wedge}(l,r) = \int_{-\infty}^{\infty} s^{\wedge}(f) \overbrace{\times\, e^{-i\pi\{\frac{N-1}{f_s}\}(f_{co,l}-f)} \frac{sin\{\pi\frac{N}{f_s}(f_{co,l}-f)\}}{sin\{\pi\frac{1}{f_s}(f_{co,l}-f)\}}}^{f_s-periodic}$$

$$\underbrace{e^{-i\pi\{\frac{Mm-1}{mf_\tau}\}(f_{fi,r}-f)} \frac{sin\{\pi\frac{Mm}{mf_\tau}(f_{fi,r}-f)\}}{sin\{\pi\frac{1}{mf_\tau}(f_{fi,r}-f)\}}}_{mf_\tau-periodic} df \qquad (19)$$

$$f_{co,l} = f_\tau l \;\; with \;\; 0 \leq l \leq N-1$$
$$f_{fi,r} = f_T r \;\; with \;\; 0 \leq r \leq mM-1$$

The blurring function is obtained by substituting the δ-function for $s(f)$. It is a cruciform pattern resulting from the product of two mutually perpendicular periodic *sinc*-functions. The zero crossings around the main lobes are $2f_\tau$ and $2f_T$ apart in the f_{co} and f_{fi} directions respectively. The contour from -15 dB to -3 dB in steps of 3 dB has been transferred to Fig. 2. It can be seen that, in the original region, precisely one of the sampling points lies within the main lobe.

If this result is now compared with the result obtained from non-overlapping processing (classical Zoom transform, m = 1,) the following differences are evident:

- The period of the repetitions is half as large (here, only f_τ instead of $2f_\tau$ in Fig. 2).
- Consequently, the staircase-like approximation to the straight line $f_{fi} = f_{co}$ passes through two recurring regions of blurring, and therefore, the path projected into the original region likewise passes through the blurring function at two different places.

The effect of this is clearly shown in Fig. 1. It shows a highly flawed pattern resulting from the undersampling and from the associated "aliasing" of higher orders.

4 Effect of Weighting

If, to achieve a better result, one wishes to attenuate the side lobe levels more strongly by using one of the usual windows for the DFT, this produces the following consequences:

In the formula (17), the DFT of the rectangular window is replaced by the DFT of the window used.

The width of the main lobe of the blurring function can be regarded as being approximately twice as large (this applies to all effective standard windows, such as Hamming, Hanning, Dolph-Chebyshev).

Consequently, in accordance with the rule (18), the path of the data that are read out passes through the main lobe of the blurring function four times if there is no overlapping, and still does so twice if there is a 50 % overlap, thus still producing unusable results. This means that a further doubling of the overlapping must take place in order to ensure unambiguity and high attenuation of the side lobe level in the high resolution spectrum that is read out. The result can be seen in Fig. 3. Thus, the main statement made in this

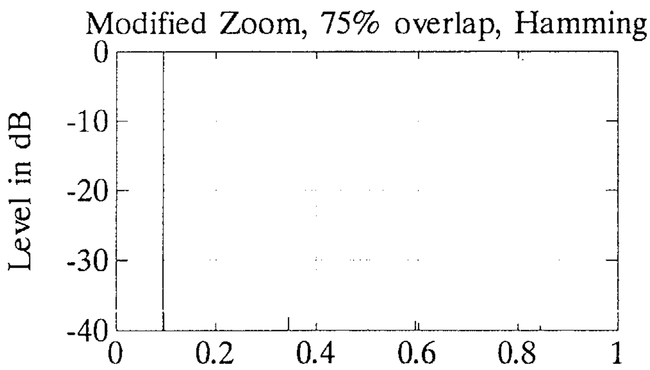

Figure 3; Frequency in Hz

chapter is as follows: A high quality, high resolution spectrum can be obtained from 75 %-overlapping, windowed short-time spectra by means of a second DFT and a read-out rule for the data (18). This modified Zoom transform avoids the disadvantages of the classical Zoom transform.

5 Summary

This report describes a method for generating a high resolution spectrum from blocks of low resolution short-time DFTs of a samples time signal. The method is based on a second DFT over the corresponding cells of the individual short-time spectra. This is known as the "Zoom transform". However, to achieve reasonable results, two modifications of the usual method are required. If attention is paid to these points, very good results are obtained from this modified Zoom transform.

6 References

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