LOSSLESS FILTER DESIGN IN TWO-BAND RATIONAL FILTER BANKS: A NEW ALGORITHM

Thierry Blu

France Télécom CNET PAB-RPE-ETP
38-40 rue du Général Leclerc
92131 Issy-Les-Moulineaux, FRANCE

email: blu@balsa.issy.cnet.fr

RÉSUMÉ

Les bancs de filtres permettant d'obtenir un échantillonnage rationnel ont été assez peu étudiés par rapport aux bancs de filtres à échantillonnage entier. C'est en particulier le cas des bancs de deux bandes dont on a pu récemment montrer que l'itération engendre des fonctions limites comme dans le cas dyadique [1]. Comme conséquence, peu d'algorithmes de synthèse de filtres existent. C'est le but de cet article que de décrire un algorithme de synthèse de filtres "sans pertes" à la fois très simple à mettre en œuvre et qui en même temps permet d'obtenir des résultats excellents en termes d'atténuation. On peut tester l'efficacité de cet algorithme dans le cas dyadique.

ABSTRACT

Filter banks with rational rate changes have not been studied much as compared to filter banks with integer rate changes. This is in particular the case of two band banks whose iteration was recently shown to generate limit functions as in the dyadic case [1]. As a consequence, few design algorithms can be found in the litterature. The purpose of this paper is the description of a design procedure of lossless filters. This new algorithm is very simple to implement, and leads to excellent results for the design parameters. This efficiency can be tested in the dyadic case.

1. INTRODUCTION

Two-band iterated filter banks (dyadic case) have become very attractive for coding purposes [6] because they decompose a signal into lower resolution subsignals. This allows to "see" the larger scale structures of the input signal as well as the smaller scale ones [9]. The notion of scale as opposed to frequency as it appears in the short-time Fourier transform, is in particular better adapted to the study of most natural signals and to the study of transients in nonstationary signals. However, for applications such as speech processing, an octave-band analysis does not seem sufficient to distinguish between the perceptual features of the signal. Usually, it is admitted that a third-of-octave band analysis is necessary for this purpose. This can be achieved with rational filter banks (RFB).

These filter banks are very similar to the dyadic ones, with the extension that in order to achieve rational rate changes, each band is built with an up-sampler as well as with a down-sampler. The figure 2 shows a global analysis-synthesis scheme for such schemes. RFBs in their general form have been studied in [2,3] and the connection between iterated RFBs with wavelet transforms have been studied in [1].

So far, very few design examples have been proposed in the rational case. In [2,3] the analogy between p-band uniform filter banks and p/q rational filter banks is exploited. Assuming paraunitarity (or losslessness) for the polyphase matrix of the system, they decompose it in paraunitary elements of degree 1, using the method in [8]. Then they use a non-linear minimisation procedure in order to find the "optimum" parameters of the decomposition. This approach is indeed very heavy since a huge number of local minima can trap the algorithm in non optimal solutions: this requires running the algorithm on different starting points.

[4,5] take another approach, which deliberately relaxes the perfect reconstruction hypothesis. Instead, the reconstruction error is minimized together with other design parameters in a two step iterative process. The first one minimizes a quadratic cost function of the synthesis parameters; the second one uses a conjugate gradient method to modify the analysis parameters so that the value of this functional decreases. At the limit of their iterative process, the filters can show a very low reconstruction error and a good frequency behaviour. An advantage of their method is that it naturally designs biothonormal filter banks, as opposed to orthonormal, or lossless filter banks. However, it is interesting to design lossless filter banks for many reasons. First, they constitute an orthonormal transform for which statistical results are simpler than in the biorthonormal case. Second, an efficient and robust implementation can be realized using the decomposition [8]. Third, only two parameters are sufficient to define an optimal polynomial, namely its degree and its transition band, since after minimizing the attenuation, the bandpass ripples are also automatically minimized.

In order to present our design method, we first state the reconstruction equations and show that it is sufficient to concentrate on the analysis low-pass polynomial $G(z)$ of the two-band scheme of figure 2. In this figure, we assume $p = q + 1$ and the paraunitary conditions

$$\tilde{G}(z) = G(z) \quad (1)$$
$$\tilde{H}(z) = H(z) \quad (2)$$

Then we shall describe the algorithm as an iterative procedure. It is important to keep in mind that the iterations are not proven to converge. However, for all the cases we tested, convergence has always been reached. In that case, we prove that the limit polynomial verifies the reconstructions relations. Moreover, it has been experimentally observed that this limit solution is totally insensitive to the initiating point. This is indeed very surprising, if we remember that the minimization problem has a huge number of minima. In fact, in the dyadic case, the algorithm selects the maximum phase polynomial.

Finally, we show some design examples, and compare with the results in the dyadic case.

2. PROBLEM FORMULATION

The analysis-reconstruction problem shown in figure 2 can be put under matricial form $\mathbf{M}(z^{-1})\mathbf{M}(z) = \mathbf{Id}$ [1]. Thanks to this equation, one checks that in the case $p - q = 1$ the knowledge of G and $\tilde{G}$ is sufficient to know H and $\tilde{H}$ up to the multiplication by a constant and a delay. In the lossless case, a simple method to find the filter H is to develop the $q \times p$ matrix $\mathbf{G}(z)$ corresponding to the filter G into paraunitary simple elements [8]

$$\mathbf{G}(z) = \mathbf{G}_0\mathbf{P}_1(z)...\mathbf{P}_N(z) \quad (3)$$

where each factor $\mathbf{P}_i$ is of the form

$$\mathbf{P}_i(z) = \mathbf{Id} + (z-1)uu^T \quad (4)$$

for some unitary vector u. The $q \times p$ matrix G_0 is then orthonormalized to give a $p \times p$ matrix $(\mathbf{G}_0^T, \mathbf{H}_0^T)^T$ whose last line will provide the filter H after multiplication by the paraunitary elements $\mathbf{P}_i$.

As in the dyadic case, an "energetic" relation (for which the proof will not be given here) can be written in the case where $p - q = 1$

$$\frac{1}{q}\sum_{k=0}^{q-1}|G(e^{\frac{2i\pi(\nu+k)}{q}})|^2 + |H(e^{2i\pi\nu})|^2 = p \quad (5)$$

This assumes of course that the filters have real coefficients. A typical tolerance scheme for G is shown in figure 1. Due to (5), the transition band on the lowpass filter G automatically imposes a transition width on H, also shown in figure 1, which is q times wider than the lowpass transition band. It is thus not necessary to bother anymore with the highpass filter H. Let

$$G(z) = \sum_{k=0}^{L} g_n z^n \quad (6)$$

2.1. RECONSTRUCTION EQUATIONS

The reconstruction relations for the lowpass filters in figure 2 can be deduced from the biorthogonality equations [1]. One finds

$$\sum_k g_{np+kq+s}\tilde{g}_{kq+s} = \delta_n \quad \text{for } s = 0..q-1 \text{ and all integer } n \quad (7)$$

where δ_n is the Kronecker symbol. An important fact is to be noticed: if the filter G is lossless the system of equations becomes twice redundant and it is sufficient to consider (7) for $n \geq 0$. This redundancy will be the basis of our algorithm.

2.2. COST FUNCTION

We fix the transition bandwidth to $\delta\nu_s$ and search for the solution of the following nonlinear minimization problem

$$\min_G \int_{\frac{1}{2p}+\delta\nu_s}^{0.5} |G(e^{2i\pi\nu})|^2 \, d\nu \quad \text{under the constraints (7)} \quad (8)$$

This problem can be further developed and leads to a set of non-decoupled quadratic equations which have a huge number of solutions, corresponding to local minima. It is thus impractical to find the solution to our problem by this method.

3. DESCRIPTION OF THE ALGORITHM

Instead of solving the exact system of equations, we can think of solving a simpler minimization problem which updates a former polynomial to a polynomial with better characteristics.

The method consists in repeating the following two operations

1. given the polynomial $G^{(s)}$ at step s, find the polynomial Γ which reaches

$$\min_\Gamma \int_{\frac{1}{2p}+\delta\nu_s}^{0.5} |\Gamma(e^{2i\pi\nu})|^2 \, d\nu \quad (9)$$

under the constraints

$$\sum_k g^{(s)}_{np+kq+s}\gamma_{kq+s} = \delta_n \quad \text{for } s = 0..q-1 \text{ and all integer } n \geq 0 \quad (10)$$

Notice that only *half* the reconstruction equations (7) are written. This in particular implies that $\Gamma, G^{(s)}$ are not a couple of biorthonormal filters.

2. let

$$G^{(s+1)} = \frac{\Gamma + G^{(s)}}{2} \quad (11)$$

These operations involve only the search for the solution of a linear system, since now (10) are linear constraints, as opposed to the quadratic constraints (7). This is very easy to implement. But iterating this, does not ensure that convergence can be achieved, that is to say

$$\lim_{s \to \infty} |G^{(s+1)} - G^{(s)}| = 0$$

We however noticed the following facts

- convergence was always reached whatever the parameters L and $\delta\nu_s$ up to the precision of the computer
- there was no dependance on the initialization for the limit polynomial
- the limit polynomial has always shown a very satisfactory frequency selectivity

If we define the reconstruction error $\epsilon^{(s)}$ to be

$$\epsilon^{(s)} = \sup_{n} \sup_{l=0..q-1} |\delta_n - \sum_k g^{(s)}_{np+kq+l} g^{(s)}_{kq+l}| \qquad (12)$$

it does not cancel in general, but we have $\lim_{s\to\infty} \epsilon^{(s)} = 0$.

4. DESIGN EXAMPLES

Due to the weak theoretical part of this algorithm, its principal interest is that it provides good filters. Due to lack of space, we give here only two convergence examples. The first one is a comparison between the well known dyadic case (Smith-Barnwell solution [7]) and the result given by the algorithm. The second one is the design of a lowpass filter of length 75 (highpass is of length 19) for $p/q = 5/4$ which corresponds to a third of octave scale factor.

Case 1: $p/q = 2/1$, length$(G) = 32$, $\delta\nu_s = 0.02$

Convergence is reached in 72 steps. In figure 3 the result of the algorithm is plotted together with the result of the corresponding Smith-Barnwell solution [7]. Only the first ripple is less attenuated (by approximately 2 or 3 dB) that the Chebyshev optimum solution. This indicates that the result provided by the algorithm is of good quality in the classical two-band case. A detailed insight into the coefficient values shows moreover that the algorithm approaches the maximum phase solution of the Chebyshev optimization problem.

Case 2: $p/q = 5/4$, length$(G) = 75$, $\delta\nu_s = 0.025$

After 70 iterations, the reconstruction error was less than 10^{-10}. The frequency response of the resulting filters are shown in figure 4. The coefficients of the lowpass and highpass filters are given in table 1. Note that even if the filter G is very long, the filter H has only 19 taps. This is a particularity of lossless rational filter banks. To achieve good selectivity for the highpass filter, one has classically to consider longer highpass filters. But these longer filters imply much longer lowpass filters (typically the length of the lowpass filter is q times the length of the highpass filter), which are expensive to design. In figure 5, the reconstruction error is plotted against the iteration step. It shows an exponential decrease by a factor of approximately 0.7. This exponential decrease has generally been observed for arbitrary (p, q), while the factor itself did not change much.

5. CONCLUSION

We have presented a new algorithm for filter design in two-band rational filter banks. The originality of the algorithm is that only half of the reconstruction equations are considered, a reduction which is valid only when the limit is achieved. In itself the algorithm is an iterative process, which is not proven to converge. Fortunately, it experimentally converges in a geometrical way, and provides good results.

6. REFERENCES

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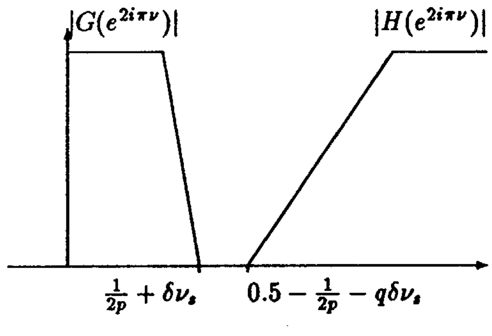

Figure 1: Tolerance schemes for the filters G and H

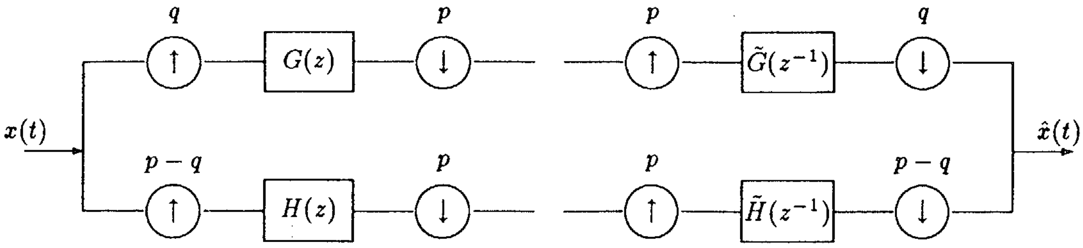

Figure 2: Analysis-synthesis two-band filter bank

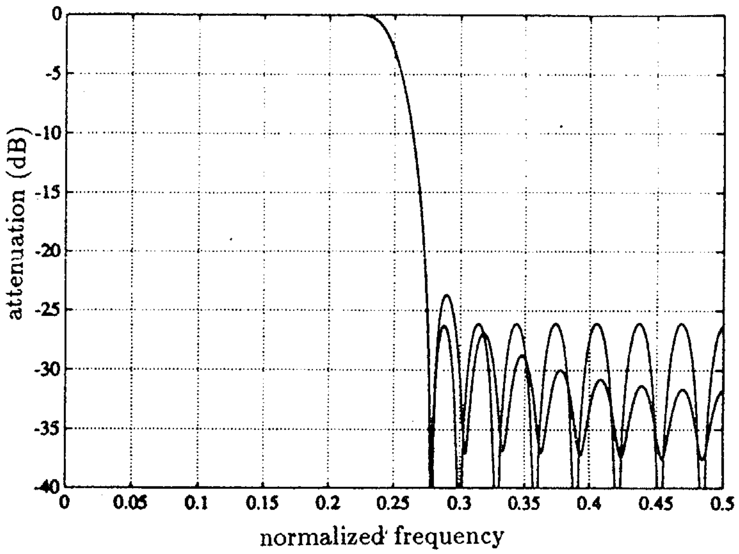

Figure 3: Algorithm vs Smith-Barnwell solution, case 1

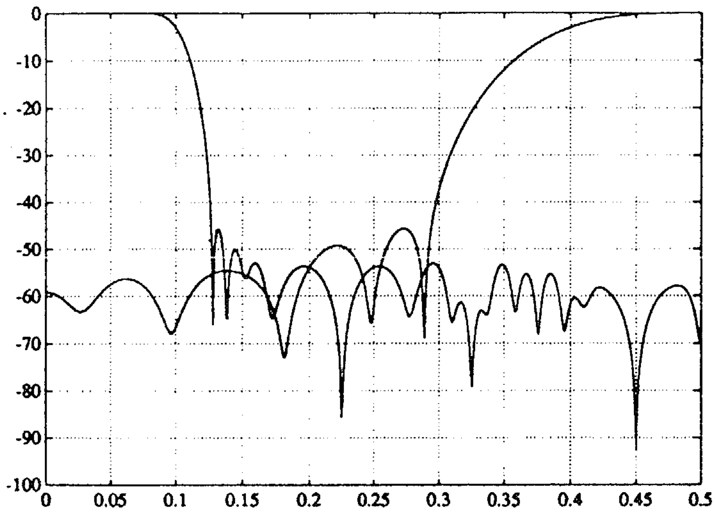

Figure 4: Lowpass and highpass filters, case 2

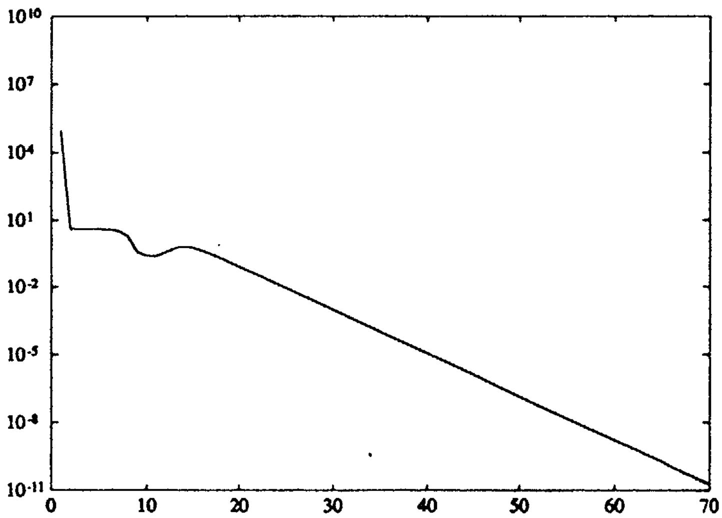

Figure 5: Reconstruction error against the iteration step, case 2

tap	lowpass filter	highpass filter	tap c'd	lowpass filter continued
0	0.0204939	-0.00176362	38	0.030226
1	0.0689152	-0.00639207	39	0.0402374
2	0.15796	0.0144022	40	0.0317526
3	0.29095	-0.00586808	41	0.013439
4	0.457521	-0.031126	42	-0.00681927
5	0.631603	0.0821973	43	-0.020554
6	0.774969	-0.101844	44	-0.0219565
7	0.846461	0.0400278	45	-0.0147118
8	0.815451	0.116426	46	-0.00330769
9	0.672149	-0.320402	47	0.00743787
10	0.436831	0.484863	48	0.0112182
11	0.155372	-0.53563	49	0.0102635
12	-0.108776	0.462957	50	0.00486268
13	-0.297992	-0.313559	51	-0.00162501
14	-0.373426	0.159974	52	-0.00405953
15	-0.329415	-0.0578897	53	-0.00524713
16	-0.193347	0.0126476	54	-0.00348715
17	-0.0149395	-0.00144908	55	0.000186182
18	0.138864	6.49091e-05	56	0.000886917
19	0.224126		57	0.00189878
20	0.221552		58	0.0017861
21	0.146318		59	-8.33974e-06
22	0.0284911		60	-0.000101617
23	-0.0815031		61	-0.000414841
24	-0.145578		62	-0.000646335
25	-0.148819		63	0
26	-0.0979612		64	4.55177e-06
27	-0.0179024		65	4.75296e-05
28	0.0561649		66	0.00014121
29	0.0983992		67	0
30	0.0964004		68	0
31	0.0591203		69	-2.12901e-06
32	0.00554392		70	-1.61789e-05
33	-0.0417218		71	0
34	-0.0648747		72	0
35	-0.058371		73	0
36	-0.0313439		74	7.24707e-07
37	0.00299839			

Table 1: Filters obtained in case 2