LEGENDRE NONUNIFORM DISCRETE FOURIER TRANSFORM AND ITS APPLICATION FOR SPECTRAL ESTIMATION

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RÉSUMÉ

Cet article introduit une nouvelle transformation de Fourier discrète de précision élevée dénommée **la transformation de Fourier discrète non-uniforme de type Legendre** (en anglais: **Legendre nonuniform discrete Fourier transform, LNDFT**). En supposant que les signaux sont réels, ayant une durée finie et aussi une bande de fréquences limitée, le modèle présent est base sur l'estimation de l'ensemble des échantillons non-uniforment prélevés de la transformée de Fourier discrète conformément aux racines du polynôme de Legendre (en définissant la **LNDFT**), dans le but d'être consistant avec une séquence donnée des échantillons uniformément prélevés dans le temps. Nous avons déduit une formule analytique pour l'inversion de la matrice complexe correspondante de type Van der Monde. A partir de la **LNDFT**, nous pouvons appliquer un interpolation utilisant les polynômes de Legendre dans le domaine de fréquence, basé sur la transformation de Legendre discrète (en anglais: **discrete Legendre transform, DLT**), pour obtenir une estimation spectrale précise et efficiente.

ABSTRACT

This paper introduces a new high accuracy discrete Fourier transform called **Legendre nonuniform discrete Fourier transform (LNDFT)**. Assuming real-valued signals that are both time-limited and also band-limited, the present model is based on the estimation of the set of nonuniformly taken samples of the Fourier transform according to Legendre polynomial roots (defining the new **LNDFT**) in order to be consistent with the given sequence of uniform time samples. An analytical formula for inversion of the corresponding complex Van der Monde matrix is deduced. Starting from the **LNDFT**, we can apply a Legendre polynomial interpolation in the frequency domain, based on the **discrete Legendre transform (DLT)**, to perform an efficient and accurate spectral estimation.

1. INTRODUCTION

There have been several approaches to define a nonuniform discrete Fourier transform [3],[4]. This paper is based on the two methods of interpolation recently proposed by Neagoe [1], [2]. In [2] a nonuniform sampling theorem for time-limited signals is given, to the aim of high accuracy preserving the integral signal characteristics (such as energy and Fourier transform); it consists of the cascade of nonuniform sampling in the time-domain according to Legendre polynomial roots, followed by the discrete Legendre transform (DLT) applied on the vector of nonuniformly taken samples. In [1] the author presents an improved interpolation method, by considering the usual uniform sampling in the time domain and the Legendre nonuniformly taken samples in the frequency domain; it leads to a high accuracy interpolation formula that minimises the instantaneous error.

This paper is an extension of a segment of [1]; it also uses some results of [2]. Assuming real-valued signals that are both time-limited and also band-limited, the present model is based on the estimation of the Legendre nonuniformly taken samples of the Fourier transform (defining the **Legendre nonuniform discrete Fourier transform = LNDFT**) in order to be consistent with the given sequence of uniform time samples. It requires to solve a linear Van der Monde system. An analytical formula for the inversion of a complex Van der Monde matrix is deduced, being useful not only for exact computation of the **LNDFT**, but also for computing a general **nonuniform discrete Fourier transform (NDFT)** as well as for other important signal processing problems. Starting from the **LNDFT**, we use a Legendre polynomial interpolation in the frequency domain, based on the **discrete Legendre transform (DLT)** in order to perform an efficient spectral estimation.

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2. THE LEGENDRE NONUNIFORM DISCRETE FOURIER TRANSFORM

Assume we have a real-valued signal, g(t), considered to be both time-limited and also band-limited $\{g : [0,n] \rightarrow \mathbf{R},\ n \in \mathbf{N},\ g \in L^2(0,n),\ G(f)=0$, for $|f|>W$, when G(f) is the Fourier transform of g(t), so that:

$$g(t)= \int_{-W}^{W} G(f)\, e^{2\pi i f t}\, dt, \qquad (2.1.)$$

$i^2= -1\}$ and also assume that the vector of uniformly taken signal samples in the time domain: $g = (g_j= g(j))_{j=0,1,\ldots,n} = (g_0\ g_1\ \ldots\ g_n)^T$ is given. An optimum interpolation formula is deduced in [1], minimising the maximum instantaneous error. It is based on the evaluation of the signal g(t) from its inverse Fourier transform (relation (2.1.)), according to the Gauss quadrature rule [1]. It leads to

$$\hat{g}_n(t)=\sum_{k=0}^{n}(w_k)^t G_k \quad , \qquad (2.2.)$$

where

$$w_k = e^{i\pi r x_k} \quad ,(r=2W; i^2=-1) \qquad (2.3.)$$

and

$$G_k=W\lambda_k\, G(f_k) \quad , (f_k=Wx_k) \qquad (2.4.)$$

λ_k being the Cristophell coefficients; x_k are the roots of the Legendre polynomial of order n+1; $G(f_k)$ are the Legendre nonuniformly taken frequency samples of the Fourier transform. The condition of convergence for interpolation is $r < 4/(\pi e) = 0.468$ (see[1]).

We define the **Legendre-nonuniform discrete Fourier transform** (**LNDFT**) as:

$$G_a= (G(f_k))_{k=0,1,\ldots,n} \qquad (2.5.)$$

or its normalised form

$$G_b= (G_k=W\, \lambda_k G(f_k))_{k=0,1,\ldots,n} \quad . \qquad (2.6.)$$

The **LNDFT** coefficients are estimated from a linear system of N=n+1 equations in order to be consistent with the given set of uniformly taken time samples **g** (see relation (2.2.) for t = j)

$$g(j) = \sum_{k=0}^{n} w_k^j G_k \quad ,(j=0,1,\ldots,n) \quad . \qquad (2.7.)$$

The above set of equations defines the **inverse Legendre Nonuniform discrete Fourier transform (Inverse LNDFT)**, that may be equivalently expressed in the matrix form

$$\mathbf{g}=\mathbf{P}\mathbf{G}_b \quad , \qquad (2.8.)$$

where **P** is the complex (n+1)×(n+1) Van der Monde matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ w_0 & w_1 & \ldots & w_n \\ \vdots & \vdots & & \vdots \\ w_0^n & w_1^n & \ldots & w_n^n \end{pmatrix} \quad , \qquad (2.9.)$$

and where w_k is given by relation (2.3.).

To deduce the **Direct LNDFT** we have to solve the above linear Van der Monde system given by (2.8.) and (2.9.). Since then, we can define the **Direct LNDFT** by the relation

$$\mathbf{G}_b=\mathbf{S}\mathbf{g} \quad , \qquad (2.10.)$$

where

$$\mathbf{S}=\mathbf{P}^{-1} \quad . \qquad (2.11.)$$

An analytical exact formula for the inversion of a complex Van der Monde matrix is deduced in the Appendix 1.

The variant $\mathbf{G}_a$ of **LNDFT** can be obtained from $\mathbf{G}_b$ as

$$G_a=(G(f_k))_{k=0,1,\ldots,n} = ((1/W\lambda_k)G_k)_{k=0,1,\ldots,n} \qquad (2.12.)$$

3. SPECTRAL ESTIMATION BASED ON LNDFT

Based on the **LNDFT** and on the results given in [2], we propose the present model of efficient spectral estimation, having the following stages:

- Starting from the vector $g=(g(j))_{j=0,1,\ldots,n}$, deduce the **direct LNDFT** expressed by the vector $\mathbf{G}_a$ (relations (2.10.), (2.11.) and (2.12.)).
- Apply the Legendre polynomial interpolation in the frequency domain, according to the formula

$$G(f)= \sum_{k=0}^{n} B_k L_k\left(\frac{f}{W}\right) \quad , \qquad (3.1.)$$

where $L_k(x)$, $x\in[-1,1]$ is the normalized Legendre polynomial of order k and where the vector $\mathbf{B}=(B_0\ B_1\ldots B_n)^t$ is given by:

$$\mathbf{B}= \Psi^{DDLT}\mathbf{G}_a \quad . \qquad (3.2.)$$

In relation (3.2.), Ψ^{DDLT} is the (n+1)×(n+1) matrix characterising the direct discrete Legendre transform introduced by Neagoe[2].

- The power spectrum density can be defined as

$$S(f)\,|_{dB}=10\log_{10}|G(f)|^2 \quad . \qquad (3.3.)$$

4. COMPUTER SIMULATION RESULTS

An example of spectral estimation according to the proposed model is given in Fig.1. Using N=16 samples only, we obtain a good quality spectral estimation (applying a sinusoidal waveform of frequency f_i, respectively the sum of two sinusoidal signals having the frequencies f_1 and f_2). The accuracy of estimating the location of the maximum can be evaluated from Fig.1. as well as from Table 1.

TABLE 1
ACCURACY OF FREQUENCY ESTIMATION
(case of a single sinusoidal waveform; N= n+1=16)

Input frequency (f_i)	Estimated frequency ($\hat{f}_i$) (location of maximum)	Relative error
0.100	0.103	3%
0.140	0.139	0.64%
0.200	0.199	0.50%

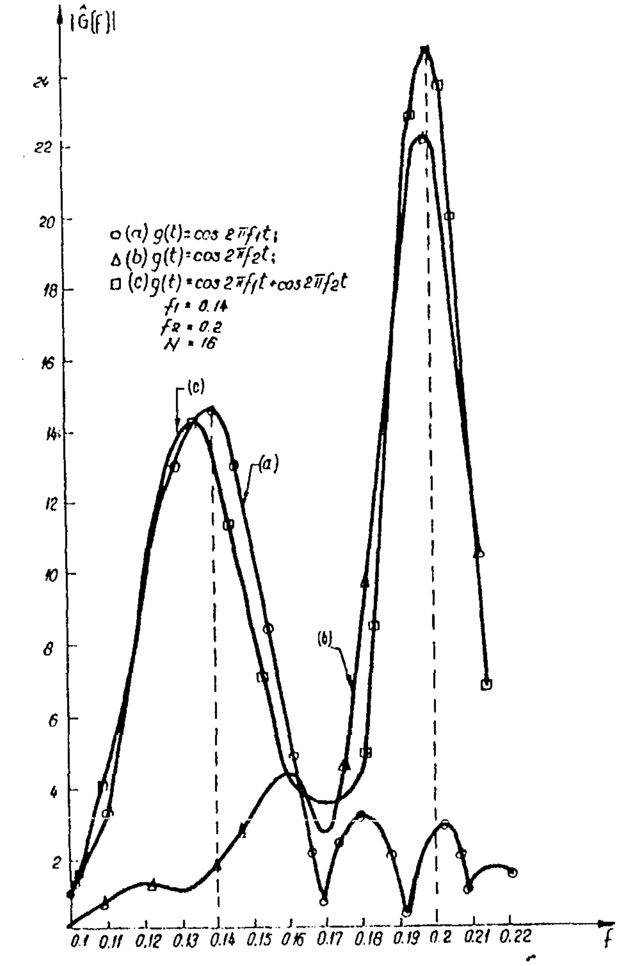

Fig. 1. Spectral estimation using **LNDFT** and **DLT** ($N=n+1=16$; $f_1=0.14$; $f_2=0.2$)

For $f_i=0.1$, the interval [0,15] corresponding to $N=n+1=16$ samples (the sampling period is $\tau = 1$) contains 3 cycles, implying a better accuracy of frequency estimation.

At the same time, the experiments must fulfil the convergence condition $r=2W< 4/(\pi e)= 0.468$; it is equivalent to $W< 0.234$ (since then, $f_i< 0.234$).

5. CONCLUDING REMARKS

1. This paper introduces a new model of discrete signal representation called the Legendre nonuniform discrete Fourier transform (**LNDFT**).

2. It is characterised by a high accuracy approximation of the correspondence between the time domain and the frequency domain (approximation of the inverse Fourier transform by the Gauss quadrature rule [1]).

3. The interpolation formula in the frequency domain is based on the fact that the set of **LNDFT** elements (vector $\mathbf{G}_a$) and the coefficients of the corresponding Legendre polynomial finite series (vector **B**) are essentially a discrete Legendre transform pair.

4. The proposed method provides a powerful tool for accurate and efficient frequency representation of signals.

5. An analytical formula for exact inversion of a complex Van der Monde matrix is deduced in the Appendix 1. It can be useful not only for **LNDFT** computation, but also for exact computation of a general **nonuniform discrete Fourier transform (NDFT)** as well as for other important signal processing problems.

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APPENDIX 1
INVERSION OF THE VAN DER MONDE MATRIX

Consider a Van der Monde matrix of order n over $\mathbb{C}$ having the form

$$\mathbf{A}=\begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{pmatrix}, a_k \in \mathbb{C} \qquad (A.1.)$$

Denote

$$\det \mathbf{A}= V_n(a_1 \dots a_n) \ . \qquad (A.2.)$$

For $1 \le k \le n$, consider the determinant

$$V_n^k(a_1,\dots,a_n)=\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{k-1} & a_2^{k-1} & \dots & a_n^{k-1} \\ a_1^{k+1} & a_2^{k+1} & \dots & a_n^{k+1} \\ \vdots & \vdots & & \vdots \\ a_1^{n} & a_2^{n} & \dots & a_n^{n} \end{vmatrix} . \qquad (A.3.)$$

For k=0, we define

$$V_n^0(a_1,\dots,a_n)=\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \dots & a_n^n \end{vmatrix}, \quad (A.4.)$$

and for k=n, we define

$$V_n^n(a_1,\dots,a_n)=\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix} . \quad (A.5.)$$

Obviously,

$$V_n^n(a_1,\dots,a_n)=V_n(a_1,\dots,a_n) . \quad (A.6.)$$

Consider, the Van der Monde determinant of order (n+1), i. e.,

$$V_{n+1}(a_1,a_2,\dots,a_n,z)=\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ a_1 & a_2 & \dots & a_n & z \\ a_1^2 & a_2^2 & \dots & a_n^2 & z^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_1^n & a_2^n & \dots & a_n^n & z^n \end{vmatrix}, \quad (A.7.)$$

where z is a complex variable.

We can easily prove that

$$V_{n+1}(a_1,a_2,\dots,a_n,z)=V_n(a_1,a_2,\dots,a_n)\cdot\prod_{i=1}^{n}(z-a_i) . \quad (A.8.)$$

On the other side, if we develop $V_{n+1}(a_1,a_2,\dots,a_n,z)$ after its last column, we yield

$$V_{n+1}(a_1,a_2,\dots,a_n,z)=V_n^{\,n}(a_1,\dots,a_n)z^n - V_n^{\,n-1}(a_1,\dots,a_n)z^{n-1}+\dots$$

$$\dots+(-1)^n V_n^{\,0}(a_1,\dots,a_n) . \quad (A.9.)$$

Consider the symmetrical polynomials of degree k, $0 \le k \le n$, having as variables $a_1,\dots,a_n$, namely

$$\begin{cases} \sigma_0(a_1,\dots,a_n)=1 \\ \sigma_1(a_1,\dots,a_n)=a_1+\dots+a_n \\ \sigma_2(a_1,\dots,a_n)=a_1a_2+a_1a_3+\dots+a_1a_n+a_2a_3+\dots+a_2a_n+\dots+a_{n-1}a_n \\ \sigma_3(a_1,\dots,a_n)=a_1a_2a_3+a_1a_2a_4+\dots+a_{n-2}a_{n-1}a_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \sigma_n(a_1,\dots,a_n)=a_1a_2\dots a_n \end{cases} \quad (A.10)$$

Taking into account that

$$\prod_{i=1}^{n}(z-a_i)=\sigma_0(a_1,\dots,a_n)z^n-\sigma_1(a_1,\dots,a_n)z^{n-1}+\dots+(-1)^n\sigma_n(a_1,\dots,a_n), \quad (A.11.)$$

relations (A.8), (A.9), (A.10) and (A.11) lead to

$$V_n^{\,k}(a_1,\dots,a_n) = V_n(a_1,\dots,a_n)\,\sigma_{n-k}(a_1,\dots,a_n) . \quad (A.12)$$
$$(0 \le k \le n)$$

Denote by A_{ij} the algebraic complement of the element a_j^{i-1} placed on the i-th row and on the j-th column of matrix **A**. We have

$$A_{ij}=(-1)^{i+j}\begin{vmatrix} 1 & 1 & \dots & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_j & \dots & a_n \\ \vdots & \vdots & & \vdots & & \vdots \\ a_1^{i-1} & a_2^{i-1} & \dots & a_j^{i-1} & \dots & a_n^{i-1} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_j^{n-1} & \dots & a_n^{n-1} \end{vmatrix} =$$

$$=(-1)^{i+j}\,V_{n-1}^{i-1}(a_1,\dots,a_{j-1},a_{j+1},\dots,a_n) . \quad (A.13.)$$

From relation (A.12),obtain

$$A_{ij}=(-1)^{i+j}V_{n-1}(a_1,\dots,a_{j-1},a_{j+1},\dots,a_n)\sigma_{n-i}(a_1,\dots,a_{j-1},a_{j+1},\dots,a_n) . \quad (A.14.)$$

We yield

$$\frac{A_{ij}}{V_n(a_1,\dots,a_n)}=(-1)^{i+j}\frac{\sigma_{n-i}(a_1,\dots,a_{j-1},a_{j+1},\dots,a_n)}{\prod_{k=1}^{j-1}(a_j-a_k)\prod_{k=j+1}^{n}(a_k-a_j)} . \quad (A.15.)$$

Since then, the inverse matrix is

$$\mathbf{A}^{-1}=\left((-1)^{i+j}\frac{\sigma_{n-i}(a_1,\dots,a_{j-1},a_{j+1},\dots,a_n)}{\prod_{k=1}^{j-1}(a_j-a_k)\prod_{k=j+1}^{n}(a_k-a_j)}\right)^T_{\substack{i=1,\dots,n\\ j=1,\dots,n}} . \quad (A.16.)$$