FAST ALGORITHM FOR MINIMUM-NORM DIRECTION OF ARRIVAL ESTIMATOR CALCULATION

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RÉSUMÉ

Il est proposè un alogarithme simple pour le calcul de l'apprèciation spectrale de la norme minimum. Le prèsent alogarithme a un avantage signifiant en diminution des opèrations par rapport à celui standard de la norme minimum dans le cas òù le nombre de sources est assez moins important par rapport au nombre d'èlèments de la grille d'antenne.

ABSTRACT

The novel fast algorithm for minimum-norm direction of arrival estimator calculation is presented. This algorithm employes special power basis instead of eigenvector one and requires only a priori knowledge of a threshold between signal subspace and noise subspace eigenvalues. Presented algorithm provides essential computational savings when the number of multiple sources is much less then the number of sensors.

1. INTRODUCTION

The eigenstructure based direction of arrival (DOA) estimation methods have been a topic of great interest since the classic works of Pisarenko [1], Schmidt [2], Bienvenu [3], Kumaresan and Tufts [4]. The high-resolution minimum-norm estimator, proposed by Kumaresan and Tufts [4], is one of the popular methods of DOA estimation, because provides excellent performance of resolution of multiple sources in passive sensor arrays for a wide signal/noise ratio (SNR) range.

The original Kumaresan and Tufts algorithm calculates the eigendecomposition of spatial covariance matrix and employes the noise subspace projection matrix in spectral estimate. As a result, the computational loads of this algorithm are very high. Recently, Brandwood [5] designed several fast algorithms for estimation of noise subspace projection matrix without eigendecomposition, but these algorithms require a priori full knowledge of noise spatial covariance matrix.

Our paper also presents a novel noneigenvector fast algorithm for minimum-norm DOA estimator calculation. This algorithm requires only a priori knowledge of any threshold between signal subspace and noise subspace eigenvalues of spatial covariance matrix and employes special power basis instead of eigenvector one.

2. POWER BASIS

Let consider a linear array of p sensors and let q ($q < p$, assume that q is known *a priori*) multiple narrowband signals impinge on the array from DOA's $\{\theta_1, \theta_2, \ldots, \theta_q\}$. The $p \times 1$ vector of signals, received at the array, can be expressed as

$$\boldsymbol{r}(t) = \boldsymbol{A}\ \boldsymbol{s}(t) + \boldsymbol{n}(t) \tag{1}$$

where $\boldsymbol{s}(t)$ is the $q \times 1$ vector of complex signals of q wavefronts, $\boldsymbol{n}(t)$ is the $p \times 1$ vector of additive noise in sensors and $\boldsymbol{A}$ is the $p \times q$ matrix

$$\boldsymbol{A}(t) = [\boldsymbol{a}(\theta_1), \boldsymbol{a}(\theta_2), \ldots, \boldsymbol{a}(\theta_q)] \qquad (2)$$

Here $\boldsymbol{a}(\theta_i)$ is the steering vector of the array toward the direction θ_i.

Assuming that additive noises are uncorrelated with the signals and between sensors and have identical variance σ^2 in each sensor, we have, that spatial $p \times p$ covariance matrix of array outputs

$$\boldsymbol{R} = E[\boldsymbol{r}(t)\boldsymbol{r}^H(t)] = \boldsymbol{A}\boldsymbol{S}\boldsymbol{A}^H + \sigma^2\boldsymbol{I}$$

where $E[\cdot]$ denotes the expectation operator, H denotes conjugate transpose, $\boldsymbol{S} = E[\boldsymbol{s}(t)\boldsymbol{s}^H(t)]$ is the $q \times q$ matrix of signal amplitudes, $\boldsymbol{I}$ is the $p \times p$ identity matrix.

The eigendecomposition of the covariance matrix $\boldsymbol{R}$ yields

$$\boldsymbol{R} = \sum_{i=1}^{p} \lambda_i\, \boldsymbol{u}_i \boldsymbol{u}_i^H$$

where $\lambda_i, \quad i = \overline{1,p} \quad (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$ and $\boldsymbol{u}_i$ are the i-th eigenvalue and i-th corresponding eigenvector, respectively.

The following properties hold [1,2]:

1) The minimum eigenvalue of $\boldsymbol{R}$ is equal to σ^2 with multiplicity $p - q$. Then, we have

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q > \lambda_{q+1} = \\ = \lambda_{q+2} = \cdots = \lambda_p = \sigma^2 \qquad (3)$$

2) The eigenvectors corresponding to the minimum eigenvalue are ortogonal to the columns of the matrix $\boldsymbol{A}$. Namely, they are ortogonal to the steering vectors of the signals:

$$\{\boldsymbol{u}_{q+1}, \boldsymbol{u}_{q+2}, \ldots, \boldsymbol{u}_p\} \perp \{\boldsymbol{a}(\theta_1), \\ \boldsymbol{a}(\theta_2), \ldots, \boldsymbol{a}(\theta_q)\} \qquad (4)$$

Theorem 1 *If the multiplicity of the minimum eigenvalue of the matrix $\boldsymbol{R}$ is equal to $p-q$, then matrix $\boldsymbol{R}^l$ with any power l can be expanded as a linear combination of matrices of finite power basis $\{\boldsymbol{I}, \boldsymbol{R}, \ldots, \boldsymbol{R}^q\}$, i.e.*

$$\boldsymbol{R}^l = c_0^{(l)}\boldsymbol{I} + c_1^{(l)}\boldsymbol{R} + \cdots + c_q^{(l)}\boldsymbol{R}^q \qquad (5)$$

where $\{c_0^{(l)}, c_1^{(l)}, \ldots, c_q^{(l)}\}$ are the coefficients for power l.

Thus, for the arbitrary power l and the arbitrary $p \times 1$ vector $\boldsymbol{f}$ the following finite power vector basis expansion exists:

$$\boldsymbol{R}^l\boldsymbol{f} = c_0^{(l)}\boldsymbol{f} + c_1^{(l)}\boldsymbol{R}\boldsymbol{f} + \cdots + c_q^{(l)}\boldsymbol{R}^q\boldsymbol{f} \qquad (6)$$

3. REPRESENTATION OF MINIMUM-NORM SPECTRA

The minimum-norm algorithm estimates the DOA's as a locations of q highest peaks of function [6]

$$P_{MN}(\theta) = \frac{1}{\left|\boldsymbol{a}^H(\theta)\hat{\boldsymbol{U}}_N\hat{\boldsymbol{U}}_N^H\boldsymbol{e}_1\right|^2} \qquad (7)$$

where $\boldsymbol{a}(\theta)$ denotes a $p \times 1$ array steering vector toward the direction θ, $\boldsymbol{e}_1$ denotes $p \times 1$ vector with all zero elements except the first one, equal to unity, $\hat{\boldsymbol{U}}_N = [\hat{\boldsymbol{u}}_{q+1}, \hat{\boldsymbol{u}}_{q+2}, \ldots, \hat{\boldsymbol{u}}_p]$ is the $p \times p - q$ matrix, constructed with the noise subspace eigenvectors, corresponding to the smallest $p-q$ eigenvalues of sample covariance matrix:

$$\hat{\boldsymbol{R}} = \frac{1}{k}\sum_{i=1}^{k} \boldsymbol{r}(t_i)\boldsymbol{r}^H(t_i) \qquad (8)$$

where k is the total number of data snapshots. Here we ignore the constant $(\boldsymbol{e}_1^H\hat{\boldsymbol{U}}_N\hat{\boldsymbol{U}}_N^H\boldsymbol{e}_1)^2$ in the numerator of (7), because it does not alter the shape of spectra.

Theorem 2 *If λ_{thr} is the threshold between signal subspace and noise subspace eigenvalues of sample covariance matrix $\hat{\boldsymbol{R}}$, i.e. $\hat{\lambda}_{q+1} < \lambda_{thr} < \hat{\lambda}_q$, then*

$$\lim_{m\to\infty} (P_m(\theta)) = P_{MN}(\theta) \qquad (9)$$

where

$$P_m(\theta) = \frac{1}{\left|\boldsymbol{a}^H(\theta)\left(\frac{1}{\lambda_{thr}^m}\hat{\boldsymbol{R}}^m + \boldsymbol{I}\right)^{-1}\boldsymbol{e}_1\right|^2} \qquad (10)$$

So, the function $P_m(\theta)$ for any value of m is the approximate representation of the minimum-norm function $P_{MN}(\theta)$. The distinction between these functions decreases as the power m increases.

4. FAST MINIMUM-NORM ALGORITHM

Let derive the fast minimum-norm DOA estimation algorithm using the introduced approximate representation of function $P_{MN}(\theta)$. The problem is to calculate the vector

$$\left((\hat{\boldsymbol{R}}/\lambda_{thr})^m + \boldsymbol{I}\right)^{-1} \boldsymbol{e}_1$$

without the inversion of $p \times p$ matrix.

Represent the vector $\hat{\boldsymbol{R}}^{q+1}\boldsymbol{e}_1$ as the finite power expansion $\boldsymbol{Bc}$, where

$$\boldsymbol{B} = [\boldsymbol{e}_1, \hat{\boldsymbol{R}}\boldsymbol{e}_1, \dots, \hat{\boldsymbol{R}}^q\boldsymbol{e}_1] \tag{11}$$

is the $p \times q+1$ matrix, $\boldsymbol{c} = (c_0, c_1, \dots, c_q)^T$ is the $q+1 \times 1$ vector of coefficients of expansion, T denotes transpose.

The LMS solution for vector $\boldsymbol{c}$, which minimizes the norm $\|\hat{\boldsymbol{R}}^{q+1}\boldsymbol{e}_1 - \boldsymbol{Bc}\|$, is given by

$$\hat{\boldsymbol{c}} = (\boldsymbol{B}^H\boldsymbol{B})^{-1}\boldsymbol{B}^H\hat{\boldsymbol{R}}^{q+1}\boldsymbol{e}_1 \tag{12}$$

Now let find the vector $\left((\hat{\boldsymbol{R}}/\lambda_{thr})^m + \boldsymbol{I}\right)^{-1} \boldsymbol{e}_1$ as a finite power basis expansion $\boldsymbol{Bd}$, where $\boldsymbol{d}$ is the $q+1 \times 1$ vector of coefficients of expansion, which must be obtained. The following linear matrix equation must be solved

$$\left(\frac{1}{\lambda_{thr}^m}\hat{\boldsymbol{R}}^m + \boldsymbol{I}\right)^{-1} \boldsymbol{e}_1 = \boldsymbol{Bd} \tag{13}$$

The $p \times q+1$ matrix $\hat{\boldsymbol{R}}\boldsymbol{B}$ can be represented as

$$\hat{\boldsymbol{R}}\boldsymbol{B} = [\hat{\boldsymbol{R}}\boldsymbol{e}_1, \hat{\boldsymbol{R}}^2\boldsymbol{e}_1, \dots, \hat{\boldsymbol{R}}^{q+1}\boldsymbol{e}_1] \tag{14}$$

Let use the LMS approximation $\boldsymbol{B}\hat{\boldsymbol{c}}$ instead of vector $\hat{\boldsymbol{R}}^{q+1}\boldsymbol{e}_1$ in (14). Then, we have:

$$\hat{\boldsymbol{R}}\boldsymbol{B} \simeq \boldsymbol{BG} \tag{15}$$

where $\boldsymbol{G}$ is the $q+1 \times q+1$ Frobenius matrix:

$$\boldsymbol{G} = \begin{pmatrix} 0 & 0 & \dots & 0 & \hat{c}_0 \\ 1 & 0 & \dots & 0 & \hat{c}_1 \\ 0 & 1 & \dots & 0 & \hat{c}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \hat{c}_q \end{pmatrix} \tag{16}$$

From (15) it also follows, that

$$\hat{\boldsymbol{R}}^m\boldsymbol{B} \simeq \boldsymbol{BG}^m \tag{17}$$

Formula (11) yields

$$\boldsymbol{e}_1 = \boldsymbol{Bg}_1 \tag{18}$$

where $\boldsymbol{g}_1$ denotes the $q+1 \times 1$ vector with all zero elements except the first one, equal to unity. Using this formula and (17), the approximate solution of (13) can be written as

$$\boldsymbol{d} = \left(\frac{1}{\lambda_{thr}^m}\boldsymbol{G}^m + \boldsymbol{I}\right)^{-1} \boldsymbol{g}_1 \tag{19}$$

Here $\boldsymbol{I}$ denotes the $q+1 \times q+1$ identity matrix (in the above equations the dimension of this matrix was equal to $p \times p$).

Fast algorithm:

Step 1 Calculate the vectors e_1, $\hat{\boldsymbol{R}}e_1$, ..., $\hat{\boldsymbol{R}}^{q+1}e_1$.

Step 2 Calculate the vector $\hat{\boldsymbol{c}}$ using (11), (12).

Step 3 Calculate the vector $\boldsymbol{d}$ for the concrete value of parameter m using (19), (16).

Step 4 Calculate the approximate minimum-norm spectral estimate (10) using (13).

Step 5 If the accuracy of the approximation of exact minimum-norm spectral estimate (7) is low, then increase m and go to the Step 3.

To compare the computational loads of original and presented minimum-norm algorithms note, that for the low number of sources ($q \ll p$) the general loads of presented algorithm are at the Step 1 ($\simeq qp^2$ complex multiplications). So, in this case our algorithm provides a substantial saving as compared with direct eigendecomposition based algorithm, which requires more than p^3 complex multiplications.

5. SIMULATION RESULTS

It should noted, that the presented algorithm is the approximate technique because of using the finite power expansion in the sample case and the finite value of parameter m. Its accuracy was compared with the eigendecomposition based (direct) minimum-norm algorithm by computer simulations.

We assumed an uniformly spaced linear array of ten sensors with half-wavelength spacing. The number of snapshots taken was 100. We considered two uncorrelated sources with the equal power for wide SNR and locations values. Simulations show, that the accuracy of the fast minimum-norm algorithm is very high. The direct and approximate spectral estimates coincide with high precision for $m \approx 4 \dots 10$.

5. CONCLUSION

We present novel fast noneigenvector algorithm for calculation of well known Kumaresan and Tufts minimum-norm DOA estimator. Our algorithm requieres only a priori knowledge of threshold between signal and noise subspace eigenvalues of covariance matrix and is approximate. Its accuracy was examined by computer simulations. Simulation results verify high precision coincidence of eigendecomposition based and presented algotithms.

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