INFORMATION AND COMPLEXITY ON THE TIME-FREQUENCY PLANE

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ABSTRACT

Many measures have been proposed for estimating signal information content and complexity on the time-frequency plane, including moment-based measures such as the time-bandwidth product and the Shannon and Rényi entropies. When applied to a time-frequency representation from Cohen's quadratic class, the Rényi information measure conforms closely to the visually based notion of complexity that we use when inspecting time-frequency images. A detailed discussion reveals many of the desirable properties of the Rényi measure for both deterministic and random signals.

RÉSUMÉ

Il est possible d'envisager beaucoup de mesures temps-fréquence pour chiffrer l'information contenue dans un signal ou sa complexité. Celles-ci incluent par exemple des extensions de la notion de produit durée-bande ou d'entropie de Shannon mais, pour les distributions quadratiques de la classe de Cohen, il apparaît que l'information de Rényi est un outil particulèrement bien adapté. On en détaille les principales propriétés et l'utilisation qui peut en être faite pour la caractérisation de signaux certains ou aléatoires.

1. INTRODUCTION

The term *component* is ubiquitous in the literature on joint time-frequency representations (TFRs) [1]. For example, one talks of suppression of Wigner distribution (WD) cross-components, concentration and resolution of auto-components, and the property that TFRs separate signal components such as parallel chirps that overlap in both time and frequency. Very often the quality of a particular TFR is judged based on subjective criteria related to the components of the signal being analyzed. Intuitively, a component is a concentration of energy in the time-frequency plane, but this notion is difficult to translate into a quantitative concept. In fact, the concept of a signal component has never been — and may never be — clearly defined.

In this paper, rather than address the question "what is a component?" directly, we will investigate several quantitative measures of signal *complexity* and *information* on the time-frequency plane. While they do not yield direct answers regarding the locations and shapes of components, these measures are intimately related to the concept of a signal component, the connection being the intuitively reasonable assumption that signals of high complexity (and therefore high information content) must be constructed from large numbers of elementary components. Viable measures of complexity and information include the time-bandwidth product and other moments on the time-frequency plane, measures of information appropriated from probability theory by Williams *et al.* [2], and parametric techniques based on decompositions into elementary building blocks as introduced by Orr *et al.* [3].

After defining Cohen's class of TFRs in the next section, we introduce these measures in Section 3. Section 4 focuses on the many attractive properties of a measure particularly suited to time-frequency analysis, the Rényi information. Extensions, as well as several possible applications of these measures, are sketched in the Conclusions.

2. TIME-FREQUENCY REPRESENTATIONS

We will study the complexity and information content of signals indirectly via Cohen's class of TFRs, a set of quadratic operators that indicate the energy content of a signal s as a function of both time t and frequency f. A TFR $C_s(t,f)$ from Cohen's class can be expressed as[1] [1]

$$C_s(t,f) = \iiint s^*\left(u-\frac{\tau}{2}\right) s\left(u+\frac{\tau}{2}\right) \Phi(\theta,\tau) \cdot e^{j2\pi(\theta u-\theta t-\tau f)}\, du\, d\theta\, d\tau, \qquad (1)$$

where the function $\Phi(\theta,\tau)$ is called the *kernel* of the TFR. Examples of Cohen's class TFRs include the WD ($\Phi(\theta,\tau)=1$), the spectrogram (Φ = ambiguity function of the time-reversed window function), and the exponential distribution ($\Phi(\theta,\tau) = e^{-\theta^2\tau^2/\beta}$) [1].

The kernel completely determines the properties of its corresponding TFR. For example, a fixed-kernel TFR possesses the energy preservation property

$$\iint C_s(t,f)\, dt\, df = \int |s(t)|^2\, dt \qquad (2)$$

and the marginal properties

$$\int C_s(t,f)\, df = |s(t)|^2, \qquad \int C_s(t,f)\, dt = |S(f)|^2 \qquad (3)$$

provided $\Phi(\theta,0)=\Phi(0,\tau)=1\ \forall\,\theta,\tau$. (The function $S(f)$ denotes the Fourier transform of the signal $s(t)$.) We will assume throughout this paper that the signal energy is normalized to one, that is, $\int |s(t)|^2 dt = 1$.

The formulas (2) and (3) evoke an analogy between a TFR and the probability density function (PDF) of a two-dimensional random variable. This parallel has been exploited with much success in the past [1,2]. In fact, as we will see, most measures on the time-frequency plane have been borrowed directly from probability theory. However, there are two key points at which this analogy breaks down. First, because of the freedom of choice of kernel function, the TFR of a given signal is nonunique. Second, most Cohen's class TFRs are nonpositive and, therefore, cannot be interpreted strictly as densities of signal energy.[2] Nevertheless, concepts from probability theory still have considerable merit in time-frequency analysis, provided caution is exercised in their interpretation.

3. INFORMATION AND COMPLEXITY MEASURES

In this section, we will both review existing measures of time-frequency information content and derive some simple extensions.

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[1] All integrals run from $-\infty$ to ∞.

[2] While there do exist classes of positive TFRs that satisfy (2) and (3) [4], we will consider only quadratic Cohen's class TFRs in this paper.

3.1. Moment-based measures

A classical measure of signal complexity is the time-bandwidth product (TBP) $\Delta_t\Delta_f$, where Δ_t and Δ_f are the RMS duration and bandwidth of the signal, respectively. The duration is computed as

$$(\Delta_t)^2 = \int t^2 |s(t)|^2 dt - \left(\int t\, |s(t)|^2 dt\right)^2. \qquad (4)$$

The bandwidth Δ_f is defined similarly, but with t replaced by f and $s(t)$ by $S(f)$. Given a TFR $C_s(t,f)$ of the signal, the TBP is readily generalized to a second-order moment in the time-frequency plane

$$\sigma^2 = \iint (t-\bar{t})^2 (f-\bar{f})^2 C_s(t,f)\, dt\, df. \qquad (5)$$

Here $\bar{t}$ and $\bar{f}$ are the mean time and mean frequency of $C_s(t,f)$, respectively.

A drawback of both above measures is that they are inherently tied to the axes of the time-frequency plane and therefore yield large values for signals such as chirps having very compact yet slanted or curved energy concentrations. A more refined measure of RMS bandwidth is the instantaneous bandwidth [1]

$$[\Delta_f(t)]^2 = \int \left[f - \bar{f}(t)\right]^2 C_s(t,f)\, df \qquad (6)$$

around the signal's instantaneous frequency $\bar{f}(t)$. This quantity can be either computed directly from the signal as $\bar{f}(t) = \frac{1}{2\pi}\frac{d}{dt}\arg s(t)$ or estimated from $C_s(t,f)$ as its conditional mean at time t. Averaging this bandwidth over time yields a new measure of local signal energy distribution around the instantaneous frequency:

$$\gamma^2 = \iint (t-\bar{t})^2 \left[f - \bar{f}(t)\right]^2 C_s(t,f)\, dt\, df. \qquad (7)$$

Interpretation problems arise with the measures (5)–(7), since, due to the nonpositivity of Cohen's class TFRs, all can take on negative values. However, kernel constraints can be constructed to ensure that these quantities remain positive. For example, the positivity of the measure (6) is guaranteed for TFRs whose kernels satisfy the constraint $\Phi(\theta,\tau) = b(\theta\tau)$, $b''(0) = \frac{1}{4}$ [1]. Kernels satisfying this constraint are easily constructed; one example is the kernel [5]

$$\Phi_{\mathrm{DED}}(\theta,\tau) = (1+a)e^{-\theta^2\tau^2/\beta_1} - a e^{-\theta^2\tau^2/\beta_2}, \qquad (8)$$

with $\frac{1}{\beta_2} = \frac{1}{8a} + \left(1+\frac{1}{a}\right)\frac{1}{\beta_1}$. The corresponding TFR is equivalent to a "difference of exponential distributions" (DED).

A problem of more fundamental importance for these functionals is that they do not truly measure signal complexity or information content [2,3]. To demonstrate, consider the signal $s_1(t) + s_2(t+T)$ constructed from two components of compact support, and note that each of the above measures increases without bound with the separation distance T. However, once s_1 and s_2 are disjoint in time, there is clearly no increase in signal complexity or information content when they are separated further.

3.2. Information measures

More promising as measures of time-frequency complexity are measures of information borrowed from probability theory, as proposed by Williams *et al.* [2]. The celebrated Shannon entropy

$$H(C_s) = -\iint C_s(t,f)\log_2 C_s(t,f)\, dt\, df, \qquad (9)$$

applied here to a normalized TFR $C_s(t,f)$, belongs to the class of Rényi entropy measures [6]

$$R_\alpha(C_s) = \frac{1}{1-\alpha}\log_2 \iint C_s^\alpha(t,f)\, dt\, df, \qquad (10)$$

parameterized by $\alpha > 0$. (The Shannon entropy is recovered as the limit of R_α as $\alpha \to 1$.) As the passage from Shannon to Rényi entropy involves only the relaxation of the mean value property of entropy from an arithmetic to an exponential mean [6], R_α behaves much like H. In particular, these functionals can be interpreted as inverse measures of concentration or "peakiness," since by analogy to probability theory, the outcomes of random experiments governed by concentrated PDFs are relatively certain and, hence, yield little information. When applied to a TFR, we will refer to R_α as the Rényi time-frequency information measure.

Clearly, locally negative values of $C_s(t,f)$ will play havoc with the logarithm in the Shannon entropy, precluding its application to most TFRs in Cohen's class. This is not the case, however, for the Rényi entropies of integer orders $\alpha \geq 2$, all of which are real-valued. We will return to the Rényi time-frequency information measure in Section 4.

3.3. Expansion-based measures

A different approach to signal complexity estimation has been proposed by Orr. In [3], the complexity of a signal with respect to a given discrete basis is defined as essentially the Shannon entropy of the basis expansion coefficients. Since the resulting complexity value varies with the choice of basis, it is necessary to carry out a minimization over all "nice" bases to obtain the true estimate of signal complexity. Use of the TFRs of Cohen's class rather than the Gabor expansions considered in [3] circumvents to some extent the problem of optimizing over different bases, but there remains the choice of particular TFR. Fortunately, we will see in the next section that the properties of the Rényi entropy of a TFR appear relatively insensitive to the particular choice of representation.

4. RENYI INFORMATION

In this section, we will investigate some of the many interesting properties of the Rényi time-frequency information measure (10).

4.1. Choice of order α

The simplest approach to measuring information on the time-frequency plane would utilize the (always positive) spectrogram and the Shannon entropy (9). However, the fixed time-frequency resolution tradeoff and bias of the spectrogram are undesirable in some applications. A more satisfactory and general approach begins with a more general TFR from Cohen's class and finishes with a Rényi entropy calculation of order α.

As discussed above, the locally negative values of Cohen's class TFRs and the desire for a real-valued information measure preclude all but integer values $\alpha \geq 2$ from the entropy calculation. The first of these values, $\alpha = 2$, is easily ruled out, since for the important WD we have the property that $\iint W_s^2(t,f)\, dt\, df = 1$ [5], and thus $R_2(W_s) = 0$ for all signals. The next possible choice, $\alpha = 3$, is the first to yield a useful information measure [2]. Along with all Rényi entropies of odd orders α, it possesses the following asymptotic invariance to cross-components in the time-frequency plane.

Property 1 *If the auto-components of a TFR $C_s(t,f)$ are separated in the time-frequency plane such that they do not overlap with any cross-components, then as the auto-component separation distance increases, we have*

$$\iint_{\mathcal{X}} C_s^\alpha(t,f)\, dt\, df \;\to\; 0 \qquad (11)$$

for all odd $\alpha \geq 1$, where $\mathcal{X}$ denotes the region containing the cross-components.[3]

[3] *Sketch of proof:* For simplicity, assume that the signal consists of just two components with compact supports T_1 and T_2 separated by a distance T (the generalization to arbitrary signals in L^2 is straightforward). Let $A'(\theta,\tau)$ denote the ambiguity function (the 2-d Fourier transform of the WD [5]) of the cross-components of $C_s(t,f)$. It is also compactly supported along the τ (delay) axis in the ranges $\tau \in \pm[|T|, |T|+T_1+T_2]$. Note that the integral (11) equals the value of the α-fold 2-d convolution $[(A'\Phi)*(A'\Phi)*\ldots*(A'\Phi)](0,0)$, where Φ is the kernel of $C_s(t,f)$. Performing this convolution yields the constraint $T > \frac{\alpha-1}{2}(T_1+T_2)$ on the separation distance that must be satisfied for (11) to hold. Note that this requirement grows linearly with the Renyi order parameter, justifying a preference for 3rd-order Renyi information over other, larger orders.

4.2. Counting property

For signals satisfying the separation conditions of Property 1, the TFR $C_s(t,f)$ is "quasi-linear," and therefore each auto-component contributes separately to the overall $R_\alpha(C_s)$ information value. In this case, the similarity to composite PDFs of statistically independent events suggests that we should expect an additive or counting behavior from $R_\alpha(C_s)$.

Property 2 *Let $s(t)$ and $\tilde{s}(t) = s(t) + s(t+T)$ be signals with T chosen such that the conditions of Property 1 hold. Then for odd $\alpha \geq 3$, we have $R_\alpha(C_{\tilde{s}}) = R_\alpha(C_s) + 1$.*[4]

As an example of this property [2], consider the $R_3(W_s)$ information of the signal $g(t) + g(t+T)$, with g a lowpass Gaussian pulse. This information is plotted in Fig. 1 versus the separation distance T in units of Δ_t (4) for $g(t)$. (At $T = 0$, the two pulses coincide and therefore, because of the assumed energy renormalization, have the same information content as a solitary pulse.) The TBP of the signal is also plotted. It is clear from the figure that, unlike the TBP which grows without bound with T, the information measure saturates exactly one bit above the value $R_3(W_g) = 2.44$. Similar results hold for three separated copies of $g(t)$ ($\log_2 3$ bits information gain), four copies (2 bits information gain), and so on.

This counting property of the Rényi information suggests a concept of *Rényi dimension* for the WD, defined as $D_\alpha(W_s) = 2^{R_\alpha(W_s) - R_\alpha(W_g)}$. This dimension indicates — relative to a "basis" of Gaussian building blocks — the number of blocks required to "cover" the WD of a given signal. Similar measures can be derived for other TFRs and other elementary building blocks.

4.3. Phase sensitivity

The results of Fig. 1 are very appealing, but are also incomplete and unrealistic, because no modulation or phase differences were introduced between the two signal components. Figure 2 illustrates a more complete set of curves of the $R_3(W_s)$ information for the signal $g(t)\cos(\pi t/6) + g(t+T)\cos(\pi(t+T)/6 + \psi)$. Each curve corresponds to a different relative phase angle ψ between 0 and π. It is apparent from the curves that while phase changes do not affect the saturation levels of the information measure, they allow many possible trajectories between the two levels, including even trajectories where an "overestimation" of information content occurs.

To interpret these results, note that as we decrease T, the auto- and cross-components of the signal begin to overlap in the time-frequency plane so that Property 1 no longer holds. At this point, relative phase plays a key role in determining information content.

In fact, the sensitivity of the $R_3(W_s)$ measure to phase is quite reasonable, given the sensitivity of closely spaced signals to relative phase. For example, Fig. 3 shows the composite signals and their respective WDs for the offset $T = C$ and relative phases $\phi = 0$ and $\phi = \frac{\pi}{2}$. The difference in appearance is striking — clearly the components in the signal and WD at top are more separated than those on the bottom. Accordingly, the $R_3(W_s)$ informations for the two signals are 3.96 and 2.96, respectively.

Property 3 *For odd $\alpha \geq 3$, the $R_\alpha(W_s)$ time-frequency information measure is very sensitive to the relative phases of closely spaced signal components.*

4.4. Effects of smoothing

Since relative phase information is carried by the cross-components of the WD, it seems reasonable that smoothing the WD (choosing kernels other than $\Phi = 1$) would lessen the effect of relative phase on information estimates. Figure 4 shows this to be the case, by repeating the same experiment as in Fig. 2, but with a matched window spectrogram TFR rather than the WD. While the spectrogram R_3 information estimate remains phase sensitive, it climbs more swiftly to the saturation level and with a reduced overshoot than $R_3(W_s)$. The price paid for this improved performance is a signal-dependent bias of information levels compared to those estimated using the Wigner distribution.

[4]The result follows directly from Property 1 and the quasi-linearity of $C_s(t,f)$ for well-separated auto-components.

Property 4 *Smoothing reduces the sensitivity of a Rényi information estimate to relative phases between signal components.*

It is important to note that some smoothing is crucial for accurate information estimates for complicated multicomponent signals with overlapping auto- and cross-components.

4.5. Random signals

Time-frequency information estimates are also very useful for random signals; however, care must be taken not to confuse the Rényi information of the *TFR* of a random signal with the Rényi information of the *PDF* of the signal.

The $R_3(W_s)$ time-frequency information estimate provides an alternative to the signal-to-noise ratio (SNR) for signals embedded in additive noise. For example, Fig. 5 illustrates the relationship between the two for a single Gaussian pulse in white Gaussian noise. Interestingly, the sigmoidal characteristic of the information measure behaves more like our eyes and ears than the SNR: for high SNRs, it indicates that there is virtually only signal present, whereas for low (negative) SNRs, it indicates that there is virtually only noise present. Furthermore, the 0 dB SNR point (the point of equal signal and noise energies) occurs roughly midway between the two information extremes.

Property 5 *The Rényi time-frequency information measure is an interesting alternative to the SNR for signals in noise.*

4.6. Normalization

Finally, we note that the definition of Rényi entropy introduced for TFRs in [2] and utilized in this paper is subtly different from that proposed for PDFs in [6]. While the measure (10) assumes a pre-normalization of signal energy to unity, the original definition of Rényi, denoted here by R'_α, utilizes a post-normalization:

$$R'_\alpha(C_s) = \frac{1}{1-\alpha} \log_2 \frac{\iint C_s^\alpha(t,f)\, dt\, df}{\iint C_s(t,f)\, dt\, df}. \qquad (12)$$

The two measures are related by $R'_\alpha(C_s) = R_\alpha(C_s) - \log_2 \iint C_s(t,f)\, dt\, df = R_\alpha(C_s) - \log_2 \Phi(0,0) \int |s(t)|^2 dt$, and thus $R'_\alpha(C_s)$ varies with the signal energy. Due to this dependence, many key properties of $R_\alpha(C_s)$ and $R'_\alpha(C_s)$ do *not* coincide. For example, for $R'_\alpha(C_s)$, Property 2 changes from a counting property to an invariance property.

5. CONCLUSIONS

Unlike the TBP and other moment-based measures, the Rényi entropy measure shows great promise for estimating the complexity of signals via the time-frequency plane. Possible applications include adaptive transforms that minimize the complexity of the TFR, contrast measures for signals in noise, and information-theoretic distance measures between different TFRs.

This paper has not addressed the question of choosing the appropriate TFR from Cohen's class for the information calculation. In fact, other classes of TFRs, such as the positive distributions [4] (which would allow the unrestricted use of the Shannon entropy) and the affine class of time-scale distributions [5] (which contains the squared magnitude of the continuous wavelet transform) could prove more appropriate for certain classes of signals.

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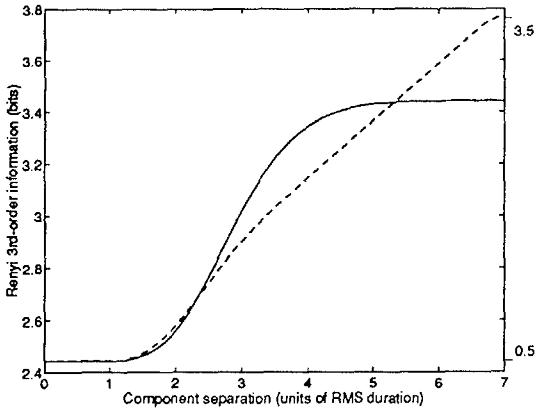


Fig. 1: $R_3(W_s)$ information of WD and TBP (dashed) vs. component separation.

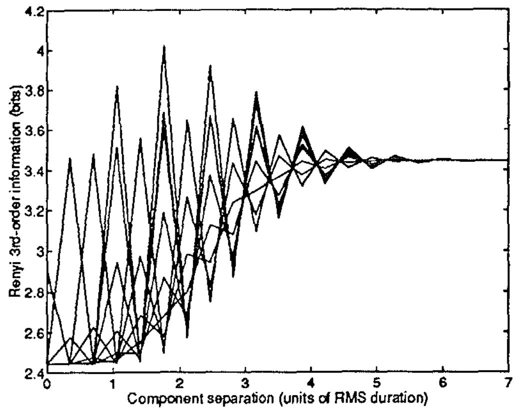
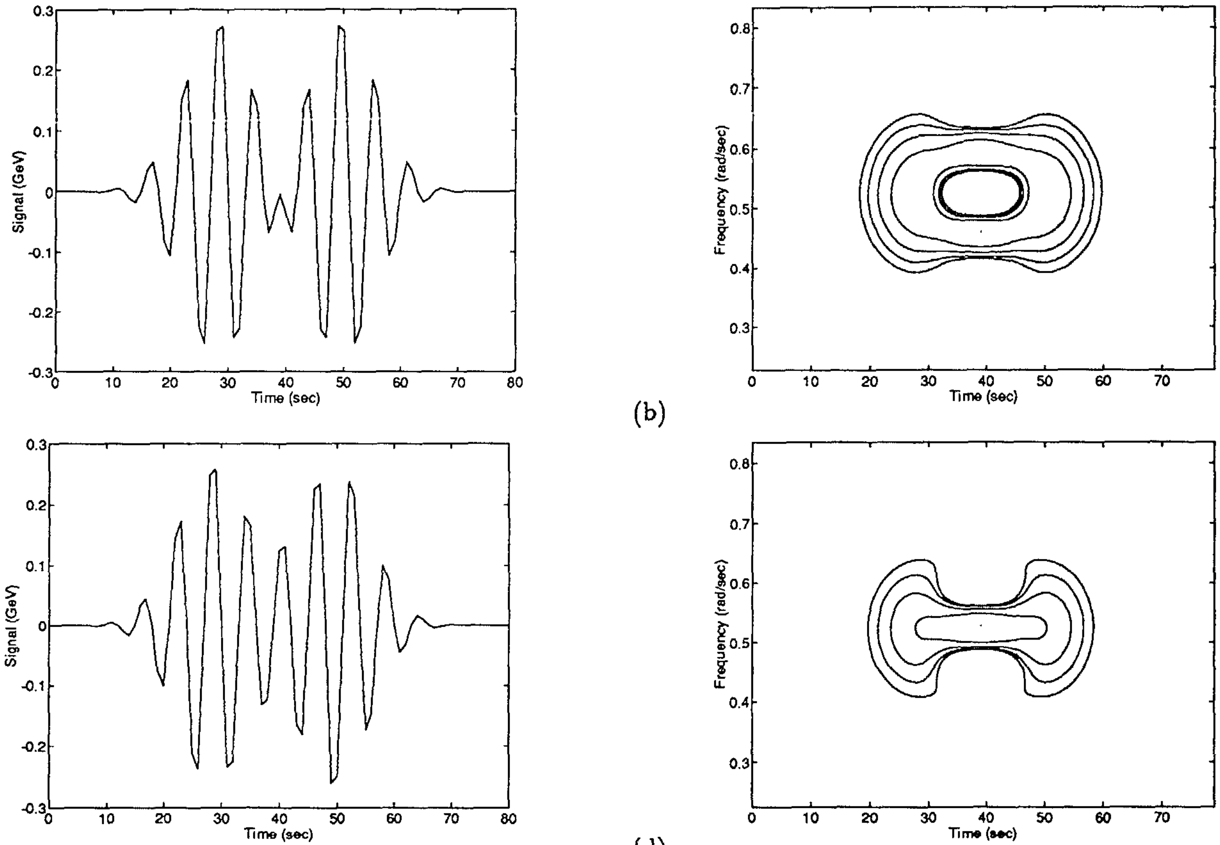


Fig. 2: $R_3(W_s)$ information of WD vs. component separation, various relative phases.



(a) (b) (c) (d)

Fig. 3: (a) Two modulated Gaussian pulses with relative phase $\psi = 0$ rad. (b) WD of signal in (a), $R_3(W_s) = 3.96$. (c) Same pulses with relative phase $\psi = \frac{\pi}{2}$ rad. (d) WD of signal in (c), $R_3(W_s) = 2.96$.

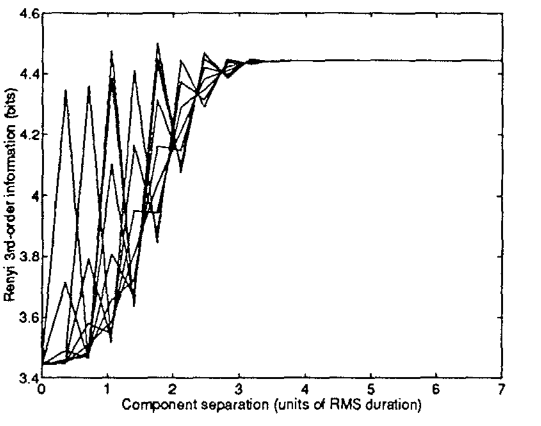


Fig. 4: $R_3(C_s)$ information of matched window spectrogram vs. component separation, various relative phases.

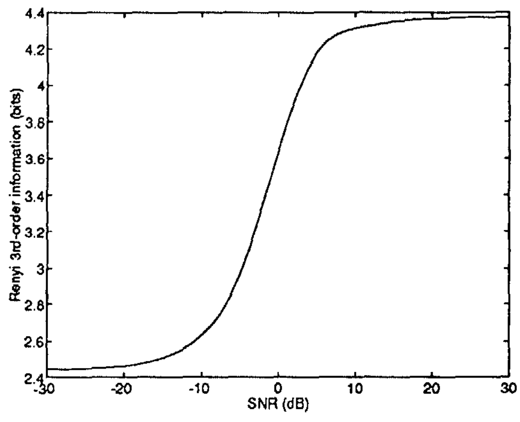


Fig. 5: $R_3(W_s)$ information of WD vs. SNR for signal + noise.