INVERTING NARROWBAND WAVELETS

by M. J. Shensa

NCCOSC RDTE DIV, code 782
San Diego, CA 92152-5000, USA
E-mail shensa@nosc.mil

RÉSUMÉ

Les transformées d'ondelettes non orthogonales discrètes jouent un rôle significatif dans le traitement du signal en permettant une résolution du temps et de l'échelle plus fine que les transformées orthogonales. Pour invertir cette transformèe, on est accoutumé à utilizer un développement en série finie des ondelettes analysantes. Bien que cette approximation convienne à beaucoups de signaux, elle manque de la précision ou exige trop d'échelles auprès des autres. Çe papier propose plusieurs algorithmes qui fournissent un meillure inverse et les compare dans le cas des ondelettes de Morlet. En même temps, des questions pratiques et théoriques sur l'inversion des transformées d'ondelettes non orthogonales sont abordées.

ABSTRACT

Discrete nonorthogonal wavelet transforms play an important role in signal processing by offering finer resolution in time and scale than their orthogonal counterparts. The standard inversion procedure for such transforms is a finite expansion in terms of the analyzing wavelet. While this approximation works quite well for many signals, it fails to achieve good accuracy or requires an excessive number of scales for others. This paper proposes several algorithms which provide more adequate inversion and compares them in the case of Morlet wavelets. In the process, both practical and theoretical issues for the inversion of nonorthogonal wavelet transforms are discussed.

I INTRODUCTION

The standard inversion procedure for discrete implementations of nonorthogonal wavelet transforms (e.g., the Morlet wavelet transform $e^{i\pi t}\, e^{-\beta^2 t^2/2}$) is a finite expansion in terms of the analyzing wavelet [1]–[2]. Formally, it is based on the theory of frames and can be thought of as a discretization of the corresponding continuous inversion formula. From this point of view several approximations are involved: the wavelet coefficients themselves, the partial sum, and the use of the analyzing wavelets rather than their duals. While a partial expansion works quite well for many (sufficiently oscillatory) signals, it fails to achieve good accuracy or requires an excessive number of scales for others. Unfortunately, the analyses found in the literature focus on frame bounds rather than the quality of of finite discrete implementations, and generally only treat relatively broadband Morlet wavelets. Since a main advantage of these wavelets over biorthogonal wavelets, which invert exactly, is their potential for increased resolution in scale, there is a clear need for a more careful examination of the invertibility of these transforms. This paper provides several alternative algorithms for inversion of the discrete wavelet transform and compares them in the case of Morlet wavelets. In the process, both practical and theoretical issues for the inversion of nonorthogonal discrete wavelet transforms are discussed.

Before proceeding with this introduction, we briefly describe our notation. $\psi(t)$ denotes the mother wavelet and is assumed to satisfy the admissibility condition $C_\psi \triangleq \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|}\, d\omega < \infty$. It is the generator of a family of decimated wavelets $\psi_{i,n} \triangleq \frac{1}{\sqrt{2^i}}\, \psi(\frac{t}{2^i} - n)$ and undecimated wavelets $\psi_n^i \triangleq \frac{1}{\sqrt{2^i}}\, \psi(\frac{t-n}{2^i})$. The sampled continuous wavelet transform of a signal s(t) is given by $W_{i,n} = \int s(t)\, \psi_{i,n}(t)\, dt$ with the undecimated version W_n^i defined analogously. Lower case letters, $w_{i,n}$ and w_n^i, denote their discrete counterpart, the discrete wavelet transform (DWT). These are typically implemented by filter banks as illustrated in Figure 1, where $[s^0]_n = s(n)$ is the discrete signal, and $\boldsymbol{f}$ and $\boldsymbol{g}$ are discrete filters. One should visualize $g_n = \overline{\psi}(-n)$ as the sampled wavelet and $\boldsymbol{f}$ as a, somewhat arbitrary, interpolation filter [3]. For details, the reader is referred to the literature ([1], [3]). Additional resolution in scale is obtained by splitting octaves into voices, $v = 0, \ldots, L-1$. That is, one implements multiple copies of filter banks, each having its filter $\boldsymbol{g}^v$ derived from a different mother wavelet $\gamma^{-v/2}\psi(\gamma^{-v}t)$ where $\gamma = 2^{1/L}$. The number of voices L should generally satisfy $L >= 1/\beta$. Although voices are essential for small β, simplicity and space dictate that we ignore them here (cf. [3], [4]).

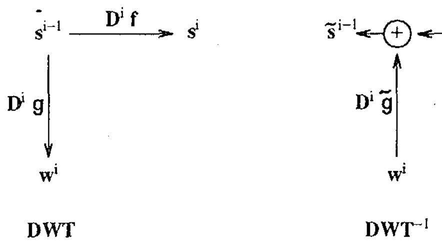

FIGURE 1. One filterbank stage of the discrete wavelet transform DWT, and of the inverse discrete wavelet transform DWT^{-1} respectively. $\mathbf{s}^0$ is the input signal and the last stage $\tilde{s}^M = s^M$.

Suppose there are M stages (octaves) to the discrete wavelet transform; i.e., $i = 0, \ldots, M-1$. Then, the DWT maps $\mathbb{R}$ into $\mathbb{R}^{M+1}$

$$\mathbf{s} \stackrel{DWT}{\longrightarrow} \{\mathbf{w}_i : i = 0, \ldots, M-1 \; ; \; \mathbf{s}^M\} \; . \qquad (1)$$

Under very weak regularity conditions the null space goes to zero as M goes to ∞. However, in order for the transformation to be nonsingular for finite M, one must include the smoothed signal $\mathbf{s}^M$ in addition to the wavelet coefficients $\mathbf{w}_i$. This transformation is clearly linear (although not time invariant in the decimated case) and may be represented by a matrix **A**. If the DWT filters **f** and **g** are finite, the transformation is locally finite (each row is zero except for a section of fixed length), but the matrix itself is infinite because the convolution acts on arbitrarily long signals. Also, the image of the transformation is a proper subset of $\mathbb{R}^{M+1}$ so that a true inverse does not exist. On the other hand, the DWT is injective so that one may "invert" objects in the range space $\mathbb{R}^{M+1}$ by specifying a left inverse satisfying $\mathbf{PA} = \mathbf{I}$. One of the most popular of these left inverses is the pseudo-inverse

$$\mathbf{P} \triangleq (\mathbf{A}^\dagger \mathbf{A})^{-1} \mathbf{A}^\dagger \qquad (2)$$

where $[\mathbf{A}^\dagger]_{ij} = \overline{\mathbf{A}}_{ji}$. More generally, if **B** is any matrix such that **BA** is nonsingular then $\mathbf{Q} = (\mathbf{B}\,\mathbf{A})^{-1}\mathbf{B}$ is also a left inverse.

Although the pseudo-inverse is a natural choice, other issues are involved. Often the error criterion is not a metric on $\mathbb{R}^{M+1}$, or computational complexity is an important consideration. Another approach, more directly related to the filterbank structure, is to invert a single stage of the DWT, thereby inverting the entire transform (cf., Figure 1). For the undecimated DWT, it suffices to find filters $\tilde{\boldsymbol{f}}$ and $\tilde{\boldsymbol{g}}$ such that

$$\tilde{\mathbf{f}} * \mathbf{f} + \tilde{\mathbf{g}} * \mathbf{g} = \boldsymbol{\delta} \qquad (3)$$

where $[\boldsymbol{\delta}]_m = \delta_{0,m}$ is the Kronecker delta [3].

The various points of view begin to merge if one considers a more general class of filters than those satisfying (3). The usual inversion procedure for nonorthogonal wavelets uses the expansion

$$s(t) \approx \sum_{n,\,i=0,\ldots,M-1} \mathrm{w}_{i,n} \widetilde{\psi_{i,n}}(t) \approx \sum_{n,\,i=0,\ldots,M-1} \mathrm{w}_{i,n} \psi_{i,n}(t) \qquad (4)$$

with the dual wavelets $\widetilde{\psi_{i,n}}$ approximated by the wavelets themselves [2]. The approximations involved in the traditional frame inverse may then be reinterpreted in a filter bank perspective. The finite sum (i.e., $M < \infty$) is equivalent to ignoring $\mathbf{s}^\mathbf{M}$, while the use of the wavelets instead of their duals is tantamount to setting $\mathbf{A}^\dagger\mathbf{A} = \mathbf{I}$ in (2). Furthermore, if we choose $\tilde{\boldsymbol{f}}$ and $\tilde{\boldsymbol{g}}$ to be $\boldsymbol{f}^\dagger \triangleq [\overline{\boldsymbol{f}}]_{-i}$ and $\boldsymbol{g}^\dagger$ respectively, then the inverse filter bank of Figure 1 computes the adjoint transformation $\mathbf{A}^\dagger = \mathbf{DWT}^\dagger$ [4]. Other suitable filter choices provide different left inverses (cf. the matrix **Q** above). In fact, we shall adopt the terminology inverse discrete wavelet transform (DWT^{-1}) to refer to any inverse filter bank whether or not the filters satisfy (3).

There are, of course, tradeoffs which accompany the various approximations. Some of the negative aspects are (a) filters satisfying equation (3) are usually prohibitively long (e.g., Morlet wavelets), (b) these filters generally provide inverses which are not the pseudo-inverse, and (c) although $\mathbf{A}^\dagger\mathbf{w}$ is readily computed, $(\mathbf{A}^\dagger\mathbf{A})^{-1}\mathbf{A}^\dagger\mathbf{w}$ is not. We find, however, in the context of inverse filter banks, that these problems may be successfully evaded. First, we note that if a true left inverse is desired, it may be computed at a moderate computational cost by iterating the forward and reverse filter banks (Neumann inverse). In fact, this technique seems much more effective than using long filters to achieve comparable accuracy. Secondly, a number of good approximate inverses exist which are sufficient for many applications without iteration. For example, a substantial improvement over (4) is obtained, at essentially no cost, by including $\mathbf{s}^M$. Also, just as the inclusion of voices provides redundancy creating a tighter frame and therefore a better inverse [2], the use of undecimated wavelets enables us to find inverses which perform better than (4) with considerably less computation.

How do we go about finding these inverses? One very effective method which lends considerable insight is to mimic the continuous case. Two such inverses shall be considered in this paper, the double integral formula [2]

$$s(t) = Re\,\frac{2}{C_\psi} \int \frac{da\,db}{a^2}\, W_b^a\, \psi_b^a(t) \; , \qquad (5)$$

and the single integral formula [5]

$$s(t) = Re\,\frac{2}{C_{1\psi}} \int_0^\infty \frac{da}{a^{3/2}}\, W_0^a(t) \qquad (6)$$

where $\psi_b^a(t) = a^{-1/2}\,\psi(\,(t-b)/a\,)$ and $C_{1\psi}$ equals $\int_{-\infty}^{\infty} |\widehat{\psi}(\omega)|/|\omega|\, d\omega$. When the scales are output by octaves, a takes on the values $a = 2^i$ for integer i. Thus, $da/a = d(\ln a) \approx \Delta(i \ln 2) = \ln 2$. In the undecimated case $db \approx \Delta b = 1$. Discretizing (5) and (6) yields

$$s(t) \approx Re\,\frac{2\ln 2}{C_\psi} \sum_{i,n} W_n^i\, \frac{1}{2^i} \psi_n^i(t) \qquad (7)$$

and

$$s(t) \approx Re\,\frac{2\ln 2}{C_{1\psi}} \sum_i \frac{1}{\sqrt{2^i}} W^i(t) \; . \qquad (8)$$

Note that the decimated version of equation (7), $s(t) \approx Re\; c \sum_{i,n} W_{i,n}\, \psi_{i,n}(t)$, is essentially the frame approximation (4).

In this paper, we shall treat four inverses: (i) the standard frame approximation (Fr) of equation (4);

(ii) the adjoint DWT (Ad) obtained by using the filters $\boldsymbol{f}^\dagger$ and $\boldsymbol{g}^\dagger$ in the inverse filter bank DWT^{-1}; (iii) the discrete analogue of the double integral (DI); and (iv) the analogue of the single integral (SI). The latter two may also be implemented by utilizing appropriate filters in the DWT^{-1}. In addition, we shall examine the performance of these (approximate) inverses under iteration; more specifically, under the Neumann formula ([2], [4]):

$$(\mathbf{BA})^{-1}\mathbf{B} = \lambda \sum_{k=0}^{\infty}(\mathbf{I} - \lambda\mathbf{BA})^k\mathbf{B} \ . \tag{9}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the DWT and DWT^{-1} and λ is a scalar sufficiently small to insure convergence. It is notable that the frame approximation, lacking the DC term (see below), is actually singular so that even with iteration low frequency parts of the signal may be lost.

II DISCRETE INVERSES

As the notation indicates, we consider DWT^{-1} to be the generic undecimated inverse discrete wavelet transform, with different left inverses corresponding to different filters $\widetilde{\boldsymbol{f}}$ and $\widetilde{\boldsymbol{g}}$. It should be emphasized that $DWT^{-1} \circ DWT \approx \boldsymbol{\delta}$ is an approximation, and that only for exceptional cases of the forward transform do there exist *finite* inverse filters which provide a true left inverse. An explicit representation of the output of the DWT^{-1} is given by (cf., Figure 1)

$$\widetilde{\mathbf{s}} = \sum_{i=0}^{M-1}\prod_{j=0}^{i-1}\left[(\mathbf{D}^j\widetilde{\boldsymbol{f}})\,*\right](\mathbf{D}^i\widetilde{\boldsymbol{g}})*\mathbf{w}^i + \prod_{j=0}^{M-1}\left[(\mathbf{D}^j\widetilde{\boldsymbol{f}})\,*\right]\mathbf{s}^M \ , \tag{10}$$

where the operator $\mathbf{D^j}$ inserts $2^j - 1$ zeros between the elements of the filter $\boldsymbol{f}$. When $\boldsymbol{g}$ is complex, s corresponds to the real part of $\widetilde{\mathrm{s}}$, that is $\mathrm{s} \approx Re\,\widetilde{\mathrm{s}}$. Note, also, that the last term contains the low frequency (DC) information.

In this section, the reader should be careful to distinguish between undecimated quantities (superscript i) and decimated quantities (subscript i) where, for example, $[\mathbf{w}_i]_n = [\mathbf{w}^i]_{2^i n}$. Although the algorithms DWT and DWT^{-1} are always undecimated, distinctions between the various inverses are more properly made by an appeal to the decimated case. For notational simplicity we present our derivations using only a single voice and replacing the constants $2\ln 2/C_\psi$ and $2\ln 2/C_{1\psi}$ by c. Space does not permit a full exposition (cf. [4]) so that we shall only derive expressions for DI. The others, SI and Ad shall merely be presented and discussed.

Let $\widetilde{\boldsymbol{f}} = \boldsymbol{\delta}/\sqrt{2}$ and $\widetilde{\boldsymbol{g}} = c\,\boldsymbol{g}^\dagger$. Abreviating the last term of (10) by DC, we have

$$\begin{aligned}
[\widetilde{\mathrm{s}}]_0 &= \sum_{i=0}^{M-1}\frac{1}{\sqrt{2^i}}\left[(\mathbf{D}^i\boldsymbol{g}^\dagger) * \mathbf{w}^i\right]_0 + DC \\
&= \sum_{i=0}^{M-1}\frac{1}{\sqrt{2^i}}\sum_n \overline{g}_n \mathrm{w}^i_{2^i n} + DC \\
&= c\sum_{i=0}^{M-1}\frac{1}{\sqrt{2^i}}\sum_n \mathrm{w}_{i,n}\overline{g}_n + DC \\
&= c\sum_{i=0}^{M-1}\sum_n \mathrm{w}_{i,n}\psi_{i,n}(0) + DC
\end{aligned} \tag{11}$$

This coincides with (4) at time $t = 0$.

Next, we extend this formula to arbitrary times $t = m$. Let $\mathbf{w}^i(m) \triangleq \mathbf{w}^i(\mathbf{T}_{-m}\mathbf{s})$ and $\mathbf{w}_i(m) \triangleq \mathbf{w}_i(\mathbf{T}_{-m}\mathbf{s})$ be the wavelet transforms obtained by first translating the signal by m points. The idea is that these transforms are computed at (i.e., centered about) time m. In this context, the frame approximation (4)

$$s_m = s(m) \approx Re\, c\sum_{i=0}^{M-1}\sum_n \mathrm{w}_{i,n}(0)\psi_{i,n}(m) \tag{12}$$

utilizes wavelet coefficients "computed" at time 0. On the other hand, since the undecimated transform and its inverse are time invariant, $\widetilde{s}_m = [\widetilde{\mathbf{T}_{-m}\mathbf{s}}]_0$. Substituting this into (11) yields

$$\widetilde{s}_m = [\widetilde{\mathbf{T}_{-m}\mathbf{s}}]_0 = c\sum_{i=0}^{M-1}\sum_n \mathrm{w}_{i,n}(m)\,\psi_{i,n}(0) + DC \ . \tag{13}$$

Note that (12) and (13) represent two quite different approximations; namely, (i) a wavelet expansion about the decimated transform computed at time 0 which uses ψ to extrapolate the signal to time m (equation (12)), and (ii) the decimated transform computed at time m, used in the wavelet expansion to approximate $s(m) = [\mathbf{T}_{-m}\mathbf{s}]_0$ at the same time as the computation (equation (13)). In addition to the DC term, we would expect (13) to be a better approximation since it does not extrapolate.

Similarly the adjoint, which may be computed by the DWT^{-1} with $\widetilde{\boldsymbol{f}} = \boldsymbol{f}^\dagger$ and $\widetilde{\boldsymbol{g}} = \boldsymbol{g}^\dagger$ [4], satisfies

$$\widetilde{s}_m \approx c\sum_{i=0}^{M-1}\frac{1}{2^i}\sum_n\sum_{r=0}^{2^i-1}\mathrm{w}_{i,n}(r)\,\psi_{i,n}(m-r) + DC \tag{14}$$

For each value of $r = 0, \ldots, 2^i - 1$, we can use (12) to estimate s_m by extrapolating from time r to time m, that is, $s_m \approx Re\sum_{i=0}^{M-1}\sum_n \mathrm{w}_{i,n}(r)\,\psi_{i,n}(m-r)$. Equation (14) is simply an average of these extrapolations. Since the longer the extrapolation, the less likely we are to have a good approximation, the quality of this result should lie somewhere in between that of (12) and (13).

Finally, we note that substituting $\widetilde{\boldsymbol{g}} = c\boldsymbol{\delta}$ and $\widetilde{\boldsymbol{f}} = \boldsymbol{\delta}/\sqrt{2}$ into (10) yields an inverse analogous to the single integration approximation (8)

$$\widetilde{s}_m = \sum_{i=0}^{M-1}\frac{c}{\sqrt{2^i}}\mathrm{w}^i_m + \frac{s^M_m}{\sqrt{2^M}} \tag{15}$$

Essentially the same formula holds for decimated wavelets; i.e., with w^i_m replaced by $\mathrm{w}_{i,n}$; however, not all octaves are available at each time. If one omits the DC term, the decimated version will provide very poor results, since at those times for which very few octaves are available up to one-half the energy may be omitted. This explains, perhaps, the reticence of many people to use the single integration approximation.

The table below summarizes the three DWT^{-1} inverses presented. For completeness, we have included the normalizations for voices.

$\mathbf{DWT^{-1}}$	$\tilde{f}$	$\tilde{g}^v$
Adjoint (Ad)	$\frac{1}{2} f^{\dagger}$	$\frac{c_2}{\gamma^v} (g^v)^{\dagger}$
Double Integral (DI)	$\frac{1}{\sqrt{2}} \delta$	$\frac{c_2}{\gamma^v} (g^v)^{\dagger}$
Single Integral (SI)	$\frac{1}{\sqrt{2}} \delta$	$\frac{c_1}{\sqrt{\gamma^v}} \delta$

III COMPARISON OF INVERSES

We very briefly compare the behavior of the inverses which have been presented. The performance of these algorithms is bound to be both wavelet (including bandwidth, number of octaves and number of voices) and signal dependent. Here, we restrict ourselves to Morlet wavelets acting on an impulsive signal. To provide a visual context, we present, in Figure 2, plots of SI, DI, and Fr. All reproduce the impulse, but with noticeable side lobes. The large dip at zero for the frame approximation (Fr) is due to the lack of a DC term. The only other obvious qualitative difference is the smaller support of the single integration (SI) formula. More may be said from an examination of the following table:

RMS errors of iterated DWT^{-1} with Morlet wavelets and impulsive signal. $\beta = 0.5$, 3 ocatves, $L = 4$, $\lambda = 0.5$.

	NUMBER OF ITERATIONS				
	0	1	5	25	50
SI	0.31	0.26	0.12	0.0038	0.000067
Ad	0.32	0.28	0.16	0.020	0.0021
DI	0.34	0.31	0.19	0.033	0.0050
Fr	0.50	0.46	0.37	0.28	0.27

Both without and with iteration, SI outperforms the other algorithms. A similar behavior seems to hold for a variety of signals and over a large range of values of β and number of voices L [4]. The lack of convergence of Fr is due to the missing DC term; however, that term is much less important for narrowband signals (cf. [4]).

IV CONCLUSIONS

Discrete nonorthogonal wavelet transforms play an important role in signal processing by providing increased resolution in time (undecimated wavelets) and scale (voices). However, as we have seen, the standard wavelet series expansion is neither the best nor, in many situations, an adequate inverse. To remedy this we proposed the inverse filter bank DWT^{-1} as a prototype inverse discrete wavelet transform. It provides a unifying framework under which various (approximate) inverses correspond to one's choice of filters. For example, the adjoint $DWT^{\dagger}$ is computed by using the adjoints of the filters from the forward transform. Under iteration it converges to the pseudo-inverse. However, the preferred inverse seems to be SI, that is, the DWT^{-1} modeled after the single integration continuous inverse. It is by far the most efficient computationally, is generally more accurate, converges more rapidly under iteration, and preserves the support of the wavelet transform.

REFERENCES

[1] J. M. Combes, A. Grossman, and P. Tchamitchian, Eds., *Wavelets: Time-Frequency Methods and Phase Space*, Berlin: Springer, IPTI, 1989.

[2] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[3] M. J. Shensa, "The Discrete Wavelet Transform: Wedding the À Trous and Mallat Algorithms," IEEE Trans. Signal Processing, Vol 40, Oct., 1992, pages 2464-2482.

[4] M. J. Shensa, "Inverting Nonorthogonal Discrete Wavelet Transforms," in preparation.

[5] N. Delprat, et al., "Asymptotic Wavelet and Gabor Analysis: Extraction of Instantaneous Frequencies," *IEEE Trans. Inf. Theory*, March, 1992, pp. 644–664.

This work was supported by ONR Code 126 and the NCCOSC IR/IED program.

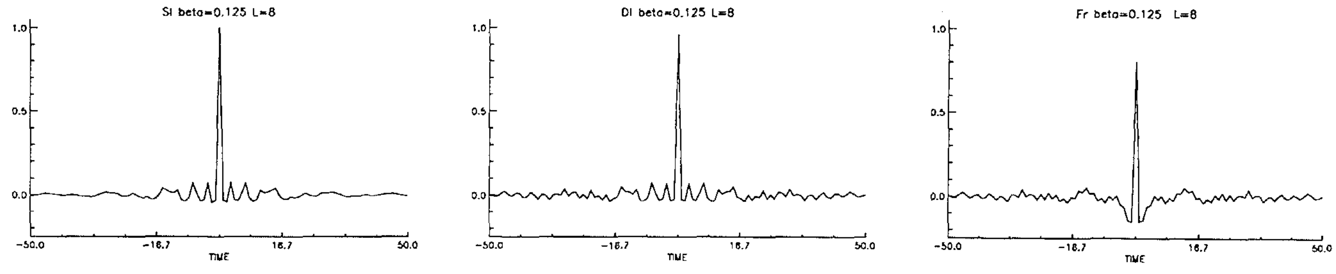

FIGURE 2. $DWT \circ DWT^{-1}$ of an impulse. The forward transform is a Morlet wavelet with $\beta = 0.125$, four octaves, and 8 voices. From left to right the inverses (without iteration) shown are SI, DI, and FR.