Formalisation mathématique et vérification numérique d'une théorie psychophysique de la discrimination de textures : la théorie des textons.

C. Lopez[1] et J.M. Morel[1]

[1]CEREMADE, Université Paris IX-Dauphine

RÉSUMÉ

Résumé. *Dans un premier temps, nous allons prouver que l'analyse multiéchelle de formes engendrée par quatre principes simples est donnée par une équation dépendant de la courbure. Ensuite, nous illustrons les avantages d'une telle analyse axiomatique en discutant une théorie psychophysique de la vision préattentive due à B. Julesz: la théorie de la discrimination de textures. Le résultat est inattendu : nous montrons que l'axiomatique de Julesz est trop bonne pour la vision humaine et conduit à un algorithme qui réalise une hyperdiscrimination.*

ABSTRACT

Abstract. *In this paper, we first prove that, under four simple axioms, the multiscale analysis of shapes is given by a curvature motion equation. In a second part, we illustrate the advantages of such an axiomatic analysis to discuss a psychophysical theory of early vision due to B. Julesz: the texture preattentive discrimination theory. The result was unexpected: We prove that the Julesz axiomatic is too good for human vision and leads to a hyperdiscrimination algorithm.*

1 Introduction.

Some theories of low level vision happen to be axiomatized. This is the case for the Julesz Texton theory, which attempts to give a formal account of how the human perception can discriminate textures in a few milliseconds. In order to confirm experimentally his theory, Julesz created pairs of simple different shapes, which where taken as building elements for creating two different but undiscriminable textures.

Axioms of shape analysis can be matched with the (not completely formal) axiomatic proposed by Julesz for his theory of preattentive texture discrimination. Therefore, the Texton theory can be numerically tested, and the result is quite different from the previous attempts and very surprising.

Indeed, the previous attempts of formalization where based on linear image multiscale filtering followed by nonlinear mechanisms compatible with physiological data. One of the most conclusive attempts in this way is due to Malik and Perona, whose algorithm matches experimentally the human performance in texture discrimination. **Now, a faithful implementation of the texton theory yields hyperdiscrimination!**

Julesz created pairs of textures which are undiscriminable for the preattentive vision but not for the algorithm deduced from his axioms. This leads to conclude that in the case of texture discrimination, the shape analysis of texton is inhibited at a scale much smaller than their size. So the axioms of texton theory (in particular the notion of Δ-neighbourhood) can be mathematically reformulated. Notice that the textual verification of a psychophysical theory has been possible because of both the axiomatization by the psychophysicist and the rigourous mathematical translation of these axioms into an algorithm.

2 The four principles of Multiscale Shape Analysis.

Define a shape or silhouette as a closed set X whose boundary is a Jordan curve of $\Re^2$. We denote by $T_t(X)$ the *shape analysed at scale t*. X is identified with its characteristic function $X(x) = 1$ if $x \in X$ and 0 else. We call multiscale analysis any family operators $(T_t)_{t\geq 0}$ acting on shapes and we set $X(t) = T_t(X)$. In order to formalize the pyramidal architecture we include $T_t = T_{t,0}$ into a family of *transition operators* $T_{t,s}$ indexed by $0 \leq s \leq t$ and satisfying

[Pyramidal architecture] $T_t = T_{t,s}T_s$ if $0 \leq s \leq t$

Assume that X and Y are two silhouettes and that for some $x \in \partial Y$ and some $r > 0$, one has $X \cap B(x, 2r) \subset Y \cap B(x, 2r)$. Assume further that the inclusion is strict in the sense that ∂X and ∂Y only meet eventually at x. Then we shall say that *the shape X is included in shape Y around x.*

[Shape local inclusion] If X is included in Y around x, then for h small enough, $T_{t+h,t}(X) \cap B(x,r) \subset T_{t+h,t}(Y) \cap B(x,r)$.

[Basic principle] Let $D = D(x, \kappa)$ be a disk with curvature κ and center x. Then $T_{t+h,t}(D)$ is a disk with radius $\rho(t, h, \kappa)$ and center x. Moreover, the function $\rho(t, h, \kappa)$ is regular.

The next optional shape preserving principle is the affine invariance of shape analysis. Set $AX(x) = X(Ax)$ for any linear map A. The affine invariance can be stated as:

[Affine inv.] For any A and $t \geq 0$, there exists a C^1 function $t'(t, A) \geq 0$ such that $AT_{t'(t,A),t'(s,A)} = T_{t,s}A$.

3 The fundamental equation of Shape Analysis.

When a point x belongs to an evolving curve, we denote by $\dot{x}$ the time derivative of x, which is a vector of $\Re^2$. By $curv(x)$ we denote the curvature of a curve which is C^2 at x.

Theorem 1. *(i) Under the four principles (pyramidal, local shape inclusion, "basic" and affine), the multiscale analysis of shapes is governed by the curvature motion equation (up to rescaling):*

$$\dot{x} = \gamma(t.curv(x))\vec{n}(x) \qquad (1)$$

where γ is defined as following: $\gamma(x) = a.x^{1/3}$ if $x \geq 0$ and $\gamma(x) = b.x^{1/3}$ if $x \leq 0$ and a, b are two nonnegative values.
(ii) If we add that $T_t(X^c) = T_t(X)^c$ [Reverse contrast invariance] then the function g in (i) is odd and $a = b$ in (ii).

Existence and regularity of the solution are proved in [10]. Sapiro and Tannenbaum also proved that this equation is geometrically equivalent to an intrinsic heat equation. It leads us to use Mackworth and Mokhtarian algorithm [8]. The first use of such an equation for shape analysis was made by Kimia [4].

We can, as well known in the "mathematical morphology school", associate with a picture u the set of its level sets

$$X_a u = \{(x, y), u(x, y) \geq a\}$$

Then, assuming that each level set is a union of silhouettes, we can simply define $T_t(u)$ as

[morphological principle] For any a, t, h and u: $X_a T_{t+h,t}(u) = T_{t+h,t}(X_a u)$

The corresponding multiscale analysis of images is governed by

$$\frac{\partial u}{\partial t} = (t.curv(x))^{1/3}|Du| \qquad (2)$$

The equation (2) has been first introduced and axiomatically justified (with a more complicated axiomatic however) in [1].

4 How to define Texton Densities. An Axiomatic Approach.

We can answer to this question by noting that the texton characteristics indicated by Julesz [3] are shape elements based on **curvature and orientation**. Furthermore, these elements are *multiscale* because of the a priori unknown size of the textons. **Therefore texton densities are nothing but multiscale curvature and orientation densities.** And, we have proved in theorem 1 that there is only one way to compute multiscale curvature and orientation.

The density of textons at scale t is given by:

$$G_\Delta(x) * (curv^+(u)(t, x))$$

$$G_\Delta(x) * (curv^-(u)(t, x))$$

where $G_\Delta(x)$ is the isotropic gaussian with variance Δ (related to the Δ-neighborhood of Julesz). By a^+ we mean $max(a, 0)$ and by a^- $max(-a, 0)$.

Then we get a formal definition of texton densities at each scale. Based on the obtained texton density channels, a segmentation of the image can be achieved by using a region growing algorithm on all the densities.

5 Conclusion.

The basic facts that we discovered by experimentation are following. The textures proposed by Julesz, as preattentively discriminable, are easily discriminated by these texton densities. Moreover, we obtain an example of hyperdiscrimination because of the non inhibition [9] of large scale curvature: in that case, we achieve shape discrimination and not texture discrimination!

Références

[1] L. Alvarez, F. Guichard, P.L. Lions and J.M. Morel. "Axiomes et equations fondamentales du traitement d'images", to appear in *Archives of Rational Mechanics and Analysis.*

[2] T. Cohignac, F. Eve, F. Guichard, C. Lopez, J.M. Morel, "Numerical Analysis of the fundamental equation of image processing". Soumis à International Journal of Computer Vision.

[3] B. Julesz, "Texton gradients: the texton theory revisited ". *Biological Cybernetics*, 54, 245-251, 1986.

[4] B.B. Kimia, "Toward a computational theory of shape. Ph. D. Dissertation", Department of Electrical Engineering, McGill University, Montreal, Canada, August 1990.

[5] B.B. Kimia, A. Tannenbaum, and S.W. Zucker, "On the evolution of curves via a function of curvature", 1: the classical case. To appear in J. of Math. Analysis and Applications.

[6] J.J. Koenderink and A.J. van Doorn, "Dynamic shape". Biol. Cyber. 53, 383-396, 1986.

[7] G. Koepfler, C. Lopez, J.M. Morel, "A multiscale algorithm for image segmentation by variational method" . Accepted at SIAM journal of Numerical Analysis.

[8] A. Mackworth and F.Mokhtarian, "A theory of multiscale, curvature-based shape representation for planar curves". IEEE Trans. Pattern Anal. Machine Intell. 14, pp 789-805, 1992.

[9] J. Malik and P. Perona, "Preattentive texture discrimination with early vision mechanisms", *Journal of the Optical Society of America*, May 1990, pp. 923-932.

[10] G. Sapiro and A. Tannenbaum, "On affine plane curve evolution", to appear in *International Journal of Computer Vision.*

[11] J. Serra, "Image analysis and mathematical morphology", Vol 1. Academic Press 1982.

[12] S.W. Zucker, "Region growing: Childhood and Adolescence (Survey)". Comp. Graphics and Image Proc. 5, 382-399, 1976.

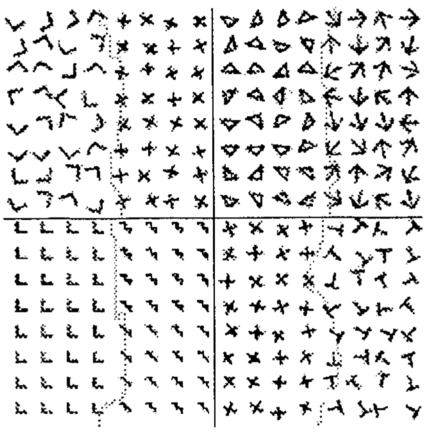

FIG. 1 - Segmentation of four texture pairs obtained by using a region growing method on the multiscale curvature densities. We only enter the number of regions and the Δ-neighbourhood.

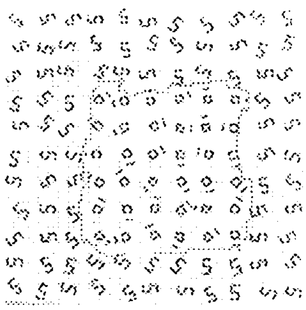

FIG. 2 - Segmentation of one texture pair by the same method.

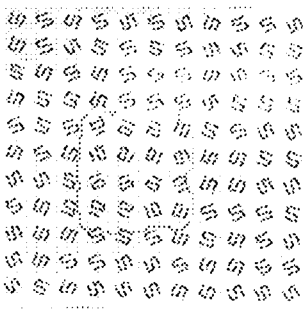

FIG. 3 - And ..., an example of hyperdiscrimination!

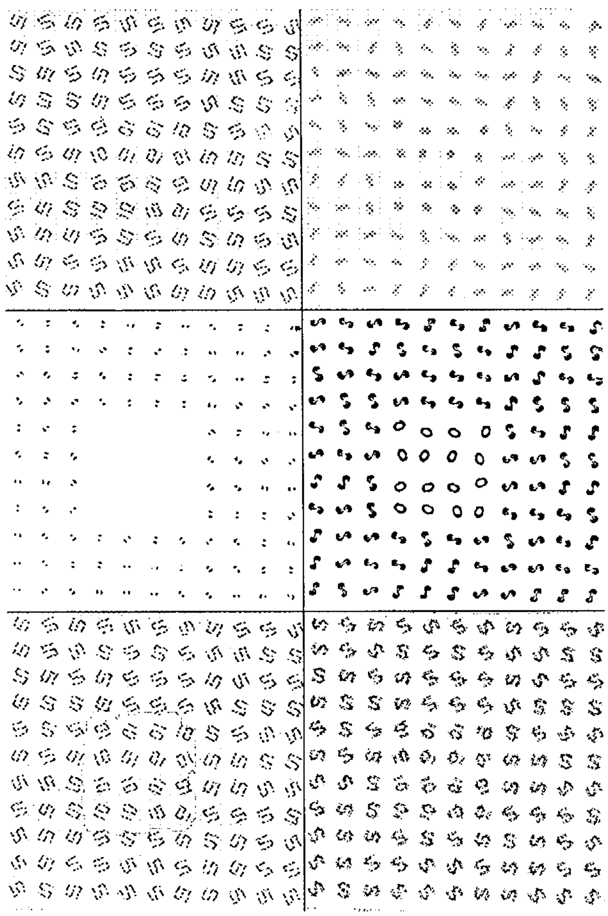

FIG. 4 - Julesz created textures which are undiscriminable for the human visual system and he thought that this undiscrimiability was a confirmation of his teory. In fact, the fundamental equation (2) allows the discrimination of that textures! First quadrant (512x512): an undiscriminable textures pair of Julesz in preattentive vision. Second quadrant: the image obtained for 7 pixels scale. Third and fourth quadrants: negative and positive curvature at 7 pixels scale. Fifth quadrant: Segmentation obtained by using a region growing method on the multiscale curvature densities. We only enter the number of regions and the Δ-neighbourhood. The last quadrant shows us that the negative curvature does not give the result; we need the *multiscale* negative and positive curvatures.