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NON-WIENER SOLUTIONS FOR THE LMS ALGORITHM -- A TIME DOMAIN APPROACH

N. J. Bershad[1] and P. L. Feintuch[2]

1. Electrical & Computer Engineering Dept., University of California, Irvine, CA, and Consultant, Hughes Aircraft Company.

2. Chief Scientist, Anti-Submarine Warfare Systems Div., Hughes Aircraft Co., Fullerton, CA.

RÉSUMÉ

Cet article présente une analyse temporelle du comportement du poids de l'algorithme LMS quand le signal d'entrée est une sinusoide déterministe. La récursion matricielle lineaire variable en fonction du temps peut être résolue exactement en utilisant une décomposition en sous-espaces orthogonaux. Nous montrons que le comportement en régime transitoire et permanent et la stabilité de l'algorithme dépendent des valeurs propres de la matrice d'état pour la récursion induite invariante dans le temps. La réponse du signal utile à la sortie du filtre est décrite par une système lineaire invariant dans les temps. Ainsi cet article présente un autre dérivation des résultats donnés dans [1] and étend les résultats de [4] au cas d'un filtre adaptatif N-tap.

ABSTRACT

This paper presents a time domain analysis of the weight behavior of the LMS algorithm when the reference input is a deterministic sinusoid. The time-varying linear matrix recursion for the LMS weight vector is solved exactly using an orthogonal subspace decomposition. The transient and steady-state behavior and the stability of the algorithm are shown to depend upon the eigenvalues of the state transition matrix for a related time-invariant recursion. The response from the desired signal to the filter output is described by a linear-time-invariant system. Thus, this paper presents an alternative derivation of the results given in [1] and extends the results in [4] to the N-tap adaptive filter case.

1. INTRODUCTION

The LMS adaptive filter algorithm is usually comprised of a tapped delay line with uniform tap spacing, a set of adjustable weights which multiply the tap outputs and a summer [2]. The weights are adjusted recursively based upon the difference between the output of the summer and an external desired signal. When the algorithm input is deterministic, the behavior of the algorithm is often quite different than when the input is a stochastic process [1,3].

These cases are denoted Non-Wiener solutions to the algorithm. This behavior was studied for adaptive noise cancelling when the reference input was a deterministic sine wave [1] and for a noise corrupted sine wave reference when the desired signal is either a sine wave or white noise in [3]. The analysis in [1] was based upon the assumption that there existed a time-invariant transfer function between the algorithm error and the adaptive filter output (although the adaptive filter weights are time-varying even in steady-state). This allowed the authors to apply Z transforms to the adaptive loop and to find an equivalent steady-state transfer function. [3] studied the case when the reference was a deterministic sine wave in white noise. [4] studied the noise-less problem for a two tap filter using a state-space approach and obtained an exact closed form solution. The solution was based upon finding a time-invariant state equation for the two tap weights. This work leads to a simple understanding of non-Wiener type adaptation but is not applicable to adaptive filters with more than two taps.

This paper studies the same problem as in [1] but uses a time-domain analysis of the adaptive filter behavior to obtain results. Transient and steady-state solutions are presented for the weights. In contrast to [1], where loop time-invariant system arguments were used, an orthogonal decomposition approach yields results in a direct manner. This alternate approach yields transient, steady-state and stability results for the weight vector. This work also extends the results in [4] to an arbitrary number of tap weights.

2. PRELIMINARIES

A. LMS Algorithm

The reference input to the tapped delay line is denoted x(n). X(n) denotes the vector of tap input values as x(n) moves down the delay line.

$X^T(n) = [\ x(n), x(n-1), x(n-2), ..., x(n-N+1)]$ (1)

N denotes the number of taps. The N-dimensional tap weight vector W(n) is defined as

$W^T(n) = [w_1(n), w_2(n), w_3(n),..... w_N(n)]$ (2)

W(n) is adjusted recursively according to [2]

$W(n+1) = W(n) + \mu\ e(n)\ X(n)$ (3)

where $e(n) = d(n)-W^T(n)X(n)$. (4)

d(n) is an external desired signal and

$$W^T(n)X(n) = \sum_{i=1}^{N} w_i(n)x(n-i+1) \quad . \qquad (5)$$

B. Algorithm Inputs

The reference input is a deterministic sinusoid whose period is an integer sub-multiple of the length of the tapped delay line. This constraint leads to significant analytical simplification and corresponds to setting the time-varying terms to zero in [1-eq. (8)]. Thus,

$x(n)=(1/2)\left[\ \exp(j\pi\ Mn/N) + \exp(-j\pi\ Mn/N)\ \right]$ (6)

where M is an integer, $0 < M < N$. x(n) has power 1/2. The vector X(n) can be written as

$X(n) = (1/2)\ [\ \exp(j\pi\ Mn/N)\ d + \exp(-j\pi\ Mn/N)d^*]$(7)

where $d^T = \{1,\exp(-j\pi M/N),\exp(-j2\pi M/N),......\exp(-j\pi(N-1)M/N)\}$. Note that d and d* are orthogonal vectors. For the steady-state part of the analysis, the desired waveform is

$d(n) = (1/2)\{\ \exp(j\pi nQ/N) + \exp\ (-j\pi nQ/N)\ \}$ (8)

where $0 < Q < N$, Q not necessarily an integer. In the transient part of the analysis, d(n) is an impulse applied at an arbitrary time.

3. ANALYSIS

A. Weight Vector Solution

Inserting eqs. (4) and (7) in eq. (3) yields

$$W(n+1) = \left[I - \frac{\mu}{4}Q(n)\right]W(n) + \mu d(n)X(n) \qquad (9)$$

where

$$Q(n) = \begin{Bmatrix} e^{j2\pi Mn/N}dd^T + d*d^T \\ +dd^T* + e^{-j2\pi Mn/N}d*d^T* \end{Bmatrix} \qquad (10)$$

The orthonormal eigenvectors of the time-varying operator $S(n) = I - (\mu/4)\ Q(n)$ are

$\phi_1(n)=[\exp\{j\pi Mn/N\}d+\exp\{j\pi Mn/N\}d^*]/(2N)^{1/2}$

$\phi_2(n) =[\exp\{j\pi Mn/N\}d-\exp\{-j\pi Mn/N\}d^*]/\ (-2N)^{1/2}$

(11)

and any set of N-2 orthonormal vectors which are also orthogonal to $\phi_1(n)$, $\phi_2(n)$. The first eigenvalue is $\lambda_1=(1-\mu N/2)$ and the remaining N-1 eigenvalues are all unity. The input in eq. (9), X(n), projects only onto $\phi_1(n)$ and the recursion update S(n) projects only onto the subspace spanned by $\phi_1(n)$ and $\phi_2(n)$. Thus, when W(0)=0, W(n) also must lie in that subspace. Therefore, the complete solution to eq. (9) is a linear combination of the first two eigenvectors,

$$W(n) = [k_1(n) + k_2(n)]\ de^{j\pi Mn/N} + [k_1(n) - k_2(n)]\ d^* e^{-j\pi Mn/N} \qquad (12)$$

Inserting eq.(12) in (9) and using the orthogonality condition on the eigenvectors, one obtains a pair of coupled recursions in $k_1(n)$ and $k_2(n)$,

$$[k_1(n+1) + k_2(n+1)] = [1 - \mu N/2]k_1(n)e^{-j\pi M/N} + k_2(n)e^{-j\pi M/N} + \frac{\mu}{2}d(n)e^{-j\pi M/N}$$

$$[k_1(n+1) - k_2(n+1)] = [1 - \mu N/2]k_1(n)e^{j\pi M/N} - k_2(n)e^{j\pi M/N} + \frac{\mu}{2}d(n)e^{j\pi M/N} \qquad (13)$$

Adding and subtracting both equations and dividing by two yields

$$k_1(n+1) = [1 - \mu N/2]\cos(\pi M/N) - j\sin(\pi M/N)k_2(n) + \frac{\mu}{2}\cos(\pi M/N)d(n)$$

$$k_2(n+1) = -j[1 - \mu N/2]\sin(\pi M/N) + \cos(\pi M/N)k_2(n) - j\frac{\mu}{2}\sin(\pi M/N)d(n) \qquad (14)$$

Eq. (14) can be written in matrix form as follows

$$\mathbf{K}(n+1) = \mathbf{V}\mathbf{K}(n) + \mathbf{R}(n)$$

where $\mathbf{K}^T(n) = [k_1(n), k_2(n)]$,

$$\mathbf{V} = \begin{bmatrix} (1-\mu N/2)\cos(\pi M/N) & -j\sin(\pi M/N) \\ -j(1-\mu N/2)\sin(\pi M/N) & \cos(\pi M/N) \end{bmatrix}$$

$$\mathbf{R}(n) = \frac{\mu}{4}\left[e^{j\pi nQ/N} + e^{-j\pi nQ/N}\right]\begin{bmatrix} \cos(\pi M/N) \\ -j\sin(\pi M/N) \end{bmatrix} \qquad (15)$$

The behavior of the recursion in eq. (15) depends upon the eigenvalues of the time-invariant matrix **V**. The eigenvalues of **V** are given by

$$\begin{vmatrix} [1-\mu N/2]\cos(\pi M/N) - \gamma & -j\sin(\pi M/N) \\ -j[1-\mu N/2]\sin(\pi M/N) & \cos(\pi M/N) - \gamma \end{vmatrix} = 0$$

$$\gamma = (1 - \mu N/4)\cos(\pi M/N) \pm\sqrt{\left(\frac{\mu N}{4}\right)^2 \cos^2(\pi M/N) - \left(1 - \frac{\mu N}{2}\right)\sin^2(\pi M/N)} \qquad (16)$$

The transient behavior of the recursion in eq. (15) depends upon the magnitude of the eigenvalues. The smaller is the magnitude, the faster is the transient response. The fastest response occurs for the critically damped case. For small μN, the eigenvalues are complex conjugates lying near the unit circle in the complex plane at $(1-\mu N/4)\, e^{\pm j\pi M/N}$. As μN increases, the eigenvalues follow an arc of a circle until meeting on the real axis, splitting and eventually crossing the unit circle when $\mu N = 4$ (instability).

Since $\mathbf{V}$ is time-invariant, the solution to eq. (15) can be easily written as

$$\mathbf{K}(n) = \mathbf{V}^n\mathbf{K}(0) + \sum_{m=0}^{n-1} \mathbf{V}^{n-m-1}\mathbf{R}(m) \tag{17}$$

It should be noted that this recursion is similar to the recursion in [4-eq. (13)] for the N=2 case. However, eq. (17) applies to the two dimensional sub-space for an N-tap filter. Furthermore, eq. (17) is applicable for $\pi M/N$ phase shift between taps rather than for only $\pi/2$ as for the model in [4].

For finite n, the summation can be expressed in terms of the eigenvalues of $\mathbf{V}$ using the similarity transformation $\mathbf{V} = \mathbf{Q}\,\boldsymbol{\Gamma}\,\mathbf{Q}^{-1}$. $\boldsymbol{\Gamma}$ is the diagonal matrix of eigenvalues given in eq. (16) and $\mathbf{Q}$ is the matrix of the eigenvectors of $\mathbf{V}$. Note that $\mathbf{V}$ is not Hermitian and thus the eigenvectors of $\mathbf{Q}$ are not orthogonal, in general. Thus, eq. (17) becomes

$$\mathbf{K}(n) = \mathbf{Q}\Gamma^n\mathbf{Q}^{-1}\mathbf{K}(0) + \mathbf{Q}\sum_{m=0}^{n-1}\Gamma^{n-m-1}\mathbf{Q}^{-1}\mathbf{R}(m) \tag{18}$$

Without loss, assume zero initial conditions, $\mathbf{K}(0)=\underline{0}$. Eq. (18) combined with eq. (12) is the complete and general solution to eq. (9). The sums in eq. (18) can be evaluated in closed form for large n without knowledge of Q.

B. Steady-State Behavior from d(n) to the Filter Output

Assuming $|\gamma_i| < 1$, i=1, 2, the sum in eq. (17) converges as n approaches infinity (steady-state), yielding

$$\lim_{n\to\infty}\mathbf{K}(n) = \frac{\mu}{4}\lim_{n\to\infty}\sum_{m=0}^{n-1}\mathbf{V}^{n-m-1}\times \left[e^{j\pi mQ/N} + e^{-j\pi mQ/N}\right]\begin{bmatrix}\cos(\pi M/N)\\ -j\sin(\pi M/N)\end{bmatrix} \tag{19}$$

Now $\lim_{n\to\infty}\sum_{m=0}^{n-1}\mathbf{V}^{n-m-1}e^{j\pi mQ/N}$

$$= e^{j\pi nQ/N}\left[e^{j\pi Q/N}\mathbf{I} - \mathbf{V}\right]^{-1} \tag{20}$$

Thus,

$$\lim_{n\to\infty}\mathbf{K}(n) = \frac{\mu}{4}\left\{\begin{matrix} e^{j\pi nQ/N}\left[e^{j\pi Q/N}\mathbf{I} - \mathbf{V}\right]^{-1} \\ +e^{-j\pi nQ/N}\left[e^{-j\pi Q/N}\mathbf{I} - \mathbf{V}\right]^{-1}\end{matrix}\right\}\times \begin{bmatrix}\cos(\pi M/N)\\ -j\sin(\pi M/N)\end{bmatrix} \tag{21}$$

Now, the matrix inverses in eq. (21) are easily evaluated, yielding

$$\mathbf{K}(n) = \frac{\mu}{4}\begin{bmatrix} e^{j\pi Q/N}\cos(\pi M/N) - 1 \\ -j\sin(\pi M/N)e^{j\pi Q/N}\end{bmatrix} \times \frac{e^{j\pi nQ/N}}{e^{j2\pi Q/N} - (2-\frac{\mu N}{2})\cos(\pi M/N)e^{j\pi Q/N} + 1 - \frac{\mu N}{2}}$$
$$+\frac{\mu}{4}\begin{bmatrix} e^{-j\pi Q/N}\cos(\pi M/N) - 1 \\ -j\sin(\pi M/N)e^{-j\pi Q/N}\end{bmatrix} \times \frac{e^{-j\pi nQ/N}}{e^{-j2\pi Q/N} - (2-\frac{\mu N}{2})\cos(\pi M/N)e^{-j\pi Q/N} + 1 - \frac{\mu N}{2}} \tag{22}$$

Using eq. (12) and (22), the filter output is given by $\mathbf{W}^T(n)\mathbf{X}(n) = Nk_1(n) =$

$$\frac{\frac{\mu N}{4}\left[e^{j\pi Q/N}\cos(\pi M/N) - 1\right]e^{j\pi nQ/N}}{e^{j2\pi Q/N} - (2-\frac{\mu N}{2})\cos(\pi M/N)e^{j\pi Q/N} + 1 - \frac{\mu N}{2}} + \frac{\frac{\mu N}{4}\left[e^{-j\pi Q/N}\cos(\pi M/N) - 1\right]e^{-j\pi nQ/N}}{e^{-j2\pi Q/N} - (2-\frac{\mu N}{2})\cos(\pi M/N)e^{-j\pi Q/N} + 1 - \frac{\mu N}{2}} \tag{23}$$

Thus, the steady-state filter output consists of the same frequencies as contained in d(n). Hence, in steady-state, there is a linear time-invariant relationship between the sinusoidal input d(n) and the filter output. Hence, a frequency response function from d(n) to the filter output is

$$J(z) = \frac{\frac{\mu N}{2}\left[z\cos(\pi M/N) - 1\right]}{z^2 - (2-\frac{\mu N}{2})\cos(\pi M/N)z + 1 - \frac{\mu N}{2}} \tag{24}$$

Eq. (24) agrees precisely with the transfer function result in [1-eq. (17)] for a desired sinusoid power of 1/2 (C=1 in [2]).

C. Transient Behavior from d(n) to the Filter Output

The time-varying behavior of eq. (17) would suggest that the system with input d(n) and output $W^T(n)X(n)$ would undergo a time-varying behavior as the adaptive filter converges to the dynamic solution given by eq. (22). To investigate this behavior, consider the impulse response from input d(n) to output $W^T(n)X(n)$. Replacing the sinusoidal input d(n) by an unit impulse applied at time p, ($d(n) = \delta(n-p)$)

$$\mathbf{K}(n) = \frac{\mu}{2}\sum_{m=0}^{n-1} \mathbf{V}^{n-m-1}\delta(m-p)\begin{bmatrix}\cos(\pi M/N)\\ -j\sin(\pi M/N)\end{bmatrix}$$

$$= \frac{\mu}{2}\mathbf{V}^{n-p-1}\begin{bmatrix}\cos(\pi M/N)\\ -j\sin(\pi M/N)\end{bmatrix} \text{ for } 0 \le p \le n-1$$

$$= 0 \text{ otherwise} \tag{25}$$

Eq. (25) is the vector impulse response. Since $\mathbf{K}(n)$ is a discrete vector sequence, consider its Z transform

$$Z[\mathbf{K}(n] = \frac{\mu}{2}\sum_{n=p+1}^{\infty} z^{-n}\mathbf{V}^{n-p-1}\begin{bmatrix}\cos(\pi M/N)\\ -j\sin(\pi M/N)\end{bmatrix}$$

$$= \frac{\mu}{2}z^{-p}[z\mathbf{I}-\mathbf{V}]^{-1}\begin{bmatrix}\cos(\pi M/N)\\ -j\sin(\pi M/N)\end{bmatrix} \tag{26}$$

Following the same procedure as from eqs. (19)-(24), the Z transform of the filter output is given by

$$Z\left[W^T(n)X(n\right] =$$

$$\frac{\frac{\mu N}{2}[z\cos(\pi M/N)-1]z^{-p}}{z^2-(2-\frac{\mu N}{2})\cos(\pi M/N)z+1-\frac{\mu N}{2}} = z^{-p}J(z) \tag{27}$$

Since the Z transform of the impulse input is z^{-p}, it follows that the impulse response from d(n) to $W^T(n)X(n)$ is time-invariant. It is obvious that the system is linear. Hence, the steady-state transfer function derived in (24) for a sinusoidal input and the transfer function obtained by taking the Z transform of the impulse response impulse are identical. Note also that the denominator polynomial in eq. (27) is the characteristic polynomial in eq.(16) for the eigenvalues of **V**.

D. Stability

The stability of the algorithm depends on the magnitude of the eigenvalues in eq. (16). If μN is sufficiently small, the eigenvalues are complex, and

$$|\gamma_i|^2 = 1 - \mu N/2 \tag{28}$$

If μN is such that the eigenvalues are real, then the larger eigenvalue is given by

$$\gamma_1 = (1-\mu N/4)\cos(\pi M/N) \;+$$

$$\frac{\mu N}{4}\cos(\pi M/N)\sqrt{1-\frac{(1-\mu N/2)\mathrm{Tan}^2(\pi M/N)}{(\mu N/4)}} \tag{29}$$

For fixed M/N, γ_1 is monotone increasing with μ and equal to one when $\mu N = 4$. Hence, the algorithm is stable for $0 < \mu N < 4$.

4. RESULTS AND CONCLUSIONS

This paper has studied the behavior of the LMS algorithm when the reference input is a deterministic sinusoid and the desired input is a either a sinusoid or an impulse. Using time domain methods, the transient weight vector solution is obtained for arbitrary adaptation speeds.

It was shown that there exists a time-invariant transfer function between the desired input and the filter output that depends on the eigenvalues of a time-invariant matrix.

The model presented here can also be used to analyze the additive noise case [3] with fewer approximations and will be the subject of a subsequent paper.

References

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