Some results on the shape-from-shading problem

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RÉSUMÉ

Nous présentons quelques résultats sur le problème de shape-from-shading: une surface lambertienne est éclairée par des rayons lumineux distribués sur l'hémisphère nord de la sphère unité S^2_+; l'équation de Horn est alors une équation intégro-différentielle qui se présente, dans certains cas, sous la forme d'une équation de Hamilton-Jacobi. Nous étudions les problèmes d'existence et d'unicité de la solution de viscosité de cette équation combinée avec différents types de conditions aux bords qui découlent naturellement des types de bords détectés dans l'image, ainsi que pour diverses sortes de distributions de sources lumineuses. Puis, nous présentons quelques résultats numériques.

ABSTRACT

We present some results on the shape-from shading problem: a Lambertian surface is illuminated by light rays distributed on the northern hemisphere of the unit sphere S^2_+; the Horn image irradiance equation is thus an integro-differential equation which reduces, in some cases, to an Hamilton-Jacobi one. We study the problem of existence and uniqueness of the viscosity solution of this equation under different types of boundary conditions which come naturally from the types of detectable edges in the image, as well as for various distributions of light sources. Some numerical results are presented.

1 Introduction

The shape-from-shading problem which is classical in vision theory corresponds to the reconstruction of a shape (a surface) from a two dimensional image. The shape is related to the image brightness by the Horn image irradiance equation (see [6]) $R(n(x,y)) = I(x,y)$ where $I(x,y)$ denotes the brightness of the image and R the reflectance map which specifies the surface as a function of its orientation (or unit normal) n. The reflectance depends on the reflectance properties of the surface and the number and distribution of light sources. We consider here the idealized case of a Lambertian surface where the action of a light ray on the image is simply the scalar product of this ray and n. Besides, if the light sources are infinitely away from the surface, they can be considered as elements of the northern hemisphere of the unit sphere namely of $S^2_+ = \{\omega \in \mathbb{R}^3 : |\omega| = 1, \omega_3 > 0\}$. Therefore a point of the surface, denoted by $(x, u(x))$ for $x \in \mathbb{R}^2$, where u is the elevation of the shape, will be illuminated by the light source ω if $\omega \in K(x,u) = \{\omega \in S^2_+ : u(x + t\omega') < u(x) + t\omega_3, \forall t > 0\}$ (where $\omega' = (\omega_1, \omega_2) \in \mathbb{R}^2$); in other ways if no other part of the surface blocks the light ray of direction ω that goes trough this point. Consequently, the equation may be rewritten in terms of the unknown function u as follows

$$\int_{S^2_+} (\omega_3 - \omega' \cdot \nabla u)(1 + |\nabla u|^2)^{-\frac{1}{2}} \mathbb{I}_{K(x,u)}(\omega) d\mu(\omega) = I \qquad (1)$$

where $\mathbb{I}_{K(x,u)}$ denotes the indicator function of the set $K(x,u)$ and μ is the distribution (a measure) of the light sources.

In some model cases, the equation takes the following form:

$$F(\nabla u(x)) = I(x) \qquad (2)$$

where F is a continuous function defined on $\mathbb{R}^2$. For instance, when the light is coming from a single source, say $\omega \in S^2_+$, F is defined by

$$F(p) = (\omega_3 - \omega' \cdot p)(1 + |p|^2)^{-\frac{1}{2}}, \text{ for } p \in \mathbb{R}^2. \qquad (3)$$

When equation (2) is a first-order nonlinear Hamilton-Jacobi partial differential equation, the viscosity solutions theory, due to Crandall and Lions (see for definition [4]), is a natural tool for equations in which the operator, say A, satisfies an order-preserving property that can be written as follows: $A[u](x_0) \geq A[v](x_0)$ if $(u - v)(x_0) = \sup(u - v)^+$ for all test functions u, v. This property is clearly satisfied by (2), as it is shown in [9]; roughly speaking it only states that if two surfaces, represented by the elevations u, $v \in C^1$ and lighted by the same distribution of sources, present a unique contact point x_0, then the one below receives more light at x_0 than the other, since less rays are blocked by other parts of the surface and $\nabla u(x_0) = \nabla v(x_0)$.

In the next section, we shall only be concerned with the case of a single light source, i.e. when F is given by (3). We shall introduce various boundary conditions which correspond to various types of edges in the image and present some existence and uniqueness results. Uniqueness is not true in general (even among C^1, C^2...solutions; see the work by Brooks, Chojnacki and Kozera [2]) but we can prove that, with some additional information, equation (2) together with natural boundary conditions is not an ill-posed problem (see also Oliensis [10]). Note also that no strong regularity assumption is necessary on the brightness I which can present jumps.

Section 3 will be dedicated to the case of multiple light sources. In some cases, uniqueness also holds and we shall give some examples in which (1) has the form of (2).

Finally, in section 4, we present briefly a numerical scheme and some experiments for a single light source.

2 Single light source

This section is dedicated to the study of the simpler case when the surface is illuminated by a single (distant) light source. We consider three different types of edges which lead to three different boundary conditions.

Apparent contours The first type of edges concerns apparent contours: beyond those edges, the surface cannot be seen even if it might be illuminated and we thus expect the exterior normal derivative of u to become $-\infty$ on the boundary which corresponds to the two-dimensional (x_1, x_2) projection of the apparent contour. When the light is vertical, only such contours occur in the image and equation (2) may be written

$$(1 + |\nabla u(x)|^2)^{-\frac{1}{2}} = I(x) \text{ in } \Omega \tag{4}$$

where Ω denotes the two-dimensional projection on the plane (x_1, x_2) of the part of the object which is lit so that $\partial\Omega$ is the apparent contour. We assume that Ω is smooth ($C^{1,1}$ for instance) and that u is continuous on $\overline{\Omega}$. We also assume that I is given on $\overline{\Omega}$, $0 < I \leq 1$ and I is Lipschitz continuous on $\overline{\Omega}$. We expect to have $I = 0$ and $\frac{\partial u}{\partial n} = -\infty$ on $\partial\Omega$, where n denotes the outward normal to Ω. It is shown in [9] that this condition, which comes naturally from the definition of the edge, can be simply formulated in terms of viscosity solutions by the condition that u is a subsolution of (4) on $\overline{\Omega}$.

Before we formulate general existence and uniqueness results, we wish to recall here that nonuniqueness is possible even for C^1 solutions because (4) does not depend on u and I may achieve the value 1 (see [9] and [12] for more details). This is why we shall only consider the case when $\{I = 1\} = \{\overline{x}\}$ for some $\overline{x} \in \Omega$. The next result may be generalized to the case when $\{I = 1\} \subset \Omega$ is a compact set (see [9]) as soon as some further information on the elevation of u is known on $\{I = 1\}$. Note also that some compatibility conditions are required on this additional information in order to obtain existence.

Theorem 1 *We assume that* $\{I = 1\} = \{\overline{x}\}$ *where* $\overline{x} \in \Omega$.
(i) Let u, $v \in C(\overline{\Omega})$ *be viscosity solutions of (4) in* Ω *and viscosity subsolutions of (4) on* $\overline{\Omega}$. *Then* $u - v$ *is constant on* $\overline{\Omega}$.
(ii) Let $d(x) = dist(x, \partial\Omega)$*; if* $(inf\{I(x)/d(x) = t\})^{-1}$ *is integrable at* $t = 0^+$*, then there exists a unique viscosity solution* $u \in C(\overline{\Omega})$ *of (4) in* Ω *which is a viscosity subsolution of (4) on* $\overline{\Omega}$ *and satisfies* $u(\overline{x}) = 0$.

Grazing light edges The second type of boundaries corresponds to the edges created where the light rays are grazing and the shadow begins. The boundary is the two-dimensional (x_1, x_2) projection of those grazing light edges. They can be defined as the set of points $(x_1, x_2, u(x_1, x_2))$ where $\omega_3 - \omega' \cdot \nabla u(x_1, x_2) = 0$.

We assume that the surface is illuminated by a single distant oblique light source $\omega = (-\alpha, -\beta, \gamma) = (\ell, \gamma) \in S^2_+$ and that some part of it cannot be seen. The illuminated part Ω of the surface is bounded by two edges namely an apparent contour Γ and a grazing light edge γ_0. The case of the contour created on the other side of the shadow (where it ends) will be studied in the next subsection.

Let us recall the equation to be satisfied in Ω:

$$(\ell \cdot \nabla u + \gamma)(1 + |\nabla u|^2)^{-\frac{1}{2}} = I. \tag{5}$$

Remark that we assume the same conditions on I as in the previous subsection.

It is shown in [9] that the boundary condition on γ_0 can be expressed in the viscosity sense by

$$\ell \cdot \nabla u + \gamma = 0 \text{ on } \gamma_0. \tag{6}$$

Theorem 2 *Let* u, $v \in C(\overline{\Omega})$ *be viscosity solution of (5)-(6) on* $\Omega \cup \gamma_0$ *and viscosity subsolution of (5) on* $\Omega \cap \Gamma$. *Then* $u - v$ *is constant on* $\overline{\Omega}$.

Shadow edges We now allow the surface to be such that a shadow forms. This clearly means in fact the formation of two edges: one was studied in the preceding subsection and the other corresponds to the border of the projected shadow or in other words to the curve where the shadow ends. The boundary shall thus consist of the two-dimensional (x_1, x_2) projection of this edge. Of course, inside the shadow region no information is available, but we can formulate on this boundary a condition that will allow some analysis of the reconstruction problem in the case when shadows occur. It is a nonlocal boundary condition.

The situation is the same as in the previous subsection, i.e. (5) has to hold inside the domain where the shape is illuminated. We shall consider two generic cases which should be sufficient to study more complicated ones by combining those two. They correspond to the possibility of the two different types of boundaries to meet or not.

The first case is the one when the projection of the grazing light edge and the projected shadow edge do not meet. In this case we have two distinct regions Ω and $\mathcal{O}$ where (5) holds and $I > 0$ on $\Omega \cup \mathcal{O}$. This two regions are separated by a shadow region. Then $\partial\Omega$ consists of the projection of an apparent contour (Γ) and of a grazing light egde (γ_0) that meet at two points and thus u can be determined up to a constant on $\overline{\Omega}$. We can now investigate the reconstruction of u in $\overline{\mathcal{O}}$. One part of $\partial\mathcal{O}$ is the shadow edge, say γ_0', and the other part that we shall call Γ' is the union of different types of boundaries. We suppose that Γ' consists of a finite number of apparent contours and grazing light edges which meet such that Γ' is (for instance) of class $C^{1,1}$. We already know what condition to impose on Γ'.

If γ_0' is the shadow edge, this means that, for each $x \in \gamma_0'$, there exists a unique point, denoted by T_x on γ_0 such that

$$x - T_x = \lambda_x \ell \text{ for some } \lambda_x > 0 \tag{7}$$

and thus

$$u(x) = u(T_x) - \lambda_x \gamma, \tag{8}$$

which means that u is given on γ_0'. Then uniqueness holds (modulo the prescription of u on $\{I = 1\}$).

The second case is when γ_0 and γ_0' meet typically at two points x_1 and x_2. We define, for each x on $\gamma_0' - \{x_1, x_2\}$, the point T_x as in (7). We denote by Ω the open illuminated part of the surface and $\Gamma = \partial\Omega$ is an apparent contour (It could consist of a finite number of apparent contours and grazing light edges but we present this case in order to simplify the presentation). Finally, $\partial\Omega \cap (\gamma_0 \cup \gamma_0') = \emptyset$. Then we have the following result:

Theorem 3 *Let u, $v \in C(\overline{\Omega})$ be viscosity solutions of (5)-(6) in $\Omega \cup \gamma_0$ and of (8) on γ_0', and viscosity subsolutions of (5) on $\Omega \cup \Gamma$. We assume that I is Lipschitz continuous in Ω, $0 \le I < 1$ in $\Omega - \{\overline{x}\}$ and $I(\overline{x}) = 1$ for some $\overline{x} \in \Omega$, and that u, v are Lipschitz continuous near x_1 and x_2. Then $u - v$ is constant on $\overline{\Omega}$.*

Discontinuous brightness In [13], we allow the function I to be discontinuous along a smooth curve Γ dividing the domain Ω in two subdomains Ω_1 and Ω_2 and prove a uniqueness result for equation (4) and (5). More precisely, the Lipschitz assumption on I may be replaced by

$$\forall (x,y) \in \Omega_i \times \Omega_j,\ i \le j,\ I(x) - I(y) \le \omega(|x-y|)$$
ω continuous nondecreasing function such that $\omega(0) = 0$

which allows a jump, and uniqueness still holds.

3 Distributed light sources

In this section, we give some elements for the study of the reconstruction of a shape illuminated by an arbitrary number of distant light sources. For simplicity of the presentation, we shall only consider the model case when there are neither blocked rays nor shadows and $\partial\Omega$ is the projection of an apparent contour. Thus, $I > 0$ in Ω and as

$$\forall x \in \Omega,\ \{\omega \in \mathrm{Supp}\mu :\ \omega_3 - \omega' \cdot \nabla u(x) \ge 0\} \subset K(x,u) \quad (9)$$

the equation (1) may be written

$$\int_{S^2_+} (\omega_3 - \omega' \cdot \nabla u)^+ (1 + |\nabla u|^2)^{-\frac{1}{2}} d\mu(\omega) = I \quad (10)$$

which is of the form (2) where F is bounded and Lipschitz continuous on $\mathbb{R}^2$. Note that (9) holds when u is concave, but it can also hold for more general shapes.

We shall also assume throughout this section that I is Lipschitz continuous in Ω.

Even if we add boundary conditions, the uniqueness of the viscosity (or C^1) solution is not automatic and uniqueness highly depends on the geometry of F. However, we can give a result in a non too particular case which can be somehow generalized.

We define the mean value of μ by

$$\overline{\omega} = \int_{S^2_+} \omega d\mu(\omega)$$

and we assume that $\overline{\omega} \cdot \omega \ge 0$ for all $\omega \in \mathrm{Supp}\mu$. In this case, $F(p) \le F(-\frac{\overline{\omega}'}{\overline{\omega}_3}) = |\overline{\omega}|$ and the constant $|\overline{\omega}|$ is the equivalent of 1 for the single light source case with respect to I, i.e. the "nonuniqueness zones" are the one where $I = |\overline{\omega}|$.

Now, if we assume that $I < |\overline{\omega}|$ in $\Omega - \{\overline{x}\}$ and $I(\overline{x}) = |\overline{\omega}|$ for some $\overline{x} \in \Omega$, and the following condition (which is, roughly speaking, a concavity condition which is usually assumed for Fs which do not depend on u)

$$\begin{aligned} &\forall \theta \in (0,1),\ \exists \nu(\theta) \in (0,1) \text{ s.t. } \forall p \in \mathbb{R}^2 \text{ s.t. } F(p) > 0 \\ &F(\theta p - (1-\theta)\tfrac{\overline{\omega}'}{\overline{\omega}_3}) \ge \\ &F(p) + \nu(\theta)(|\overline{\omega}| - F(p))(1 + |\theta p - (1-\theta)\tfrac{\overline{\omega}'}{\overline{\omega}_3}|^2)^{-\frac{1}{2}}, \end{aligned} \quad (11)$$

then the following result holds.

Theorem 4 *Let u, $v \in C(\overline{\Omega})$ be viscosity solutions in Ω and viscosity subsolutions on $\overline{\Omega}$ of (10). Then, with the above assumptions, $u - v$ is constant on $\overline{\Omega}$.*

We shall end this section with some nontrivial examples which fit in the frame of our theorem.

Our first example is the one when μ is invariant under all rotations around the $(0,0,1)$ axis. In that case, we immediatly observe that $\overline{\omega} = (0, 0, \overline{\omega}_3)$ and that F is spherically symmetric on $\mathbb{R}^2$. Finally (11) holds as soon as $\mathrm{Supp}\mu \subset \{\omega \in S^2_+ / \omega_3 \ge 1/\sqrt{2}\}$.

Another example is the case when μ is the uniform probability on S^2_+. In that case, we can show that $F(p) = (1 + (1 + |p|^2)^{-\frac{1}{2}})/4$ and then check easily that (11) holds.

4 Numerical results

We construct a numerical finite difference scheme in order to compute an approximation of the viscosity solution of (4). The approach, based on the dynamic programming principle, yields a monotone, stable and consistent scheme whose convergence was proved in [12] using the work of Barles and Souganidis [1]. Let us mention that a different approach, which relies on the work by Kushner and Dupuis [7] and yields the same scheme, was also developped by Oliensis and Dupuis in [11].

In section 2, the boundary was defined as the apparent contour of the object and the corresponding boundary condition was formulated in terms of viscosity solutions by the condition that u was a subsolution of (4) on $\overline{\Omega}$; let us remark that the boundary may also be a two-dimensional projection of a level curve of u; in other words, u is constant on $\partial\Omega$ and the corresponding boundary condition may be seen as a Dirichlet one, as it was done in [12]; actually, in this case, u is also a viscosity subsolution of (4) in $\overline{\Omega}$ and it is the way we expressed the boundary condition in the following numerical experiment.

Numerical scheme we define a regular lattice on Ω which size of mesh is $\Delta x \times \Delta y$. An approximation U will be given by: $\forall (i,j)$,

$$\left\{ \begin{array}{l} D_x^+ U_{ij} = \dfrac{U_{i+1j} - U_{ij}}{\Delta x},\ D_y^+ U_{ij} = \dfrac{U_{ij+1} - U_{ij}}{\Delta y} \\ D_x^- U_{ij} = \dfrac{U_{ij} - U_{i-1j}}{\Delta x},\ D_y^- U_{ij} = \dfrac{U_{ij} - U_{ij-1}}{\Delta y} \end{array} \right.$$

Let $g = (g_{ij})$ be a vector of functions from $\mathbb{R}^4$ to $\mathbb{R}$ defined by: $\forall (i,j)$, $\forall (a,b,c,d) \in \mathbb{R}^4$,

$$g_{ij}(a,b,c,d) = (1 + \max(a^+, b^-)^2 + \max(c^+, d^-)^2)^{-\frac{1}{2}} - I_{ij}.$$

Then, a numerical approximation U of (4) will satisfy:

$$g_{ij}(D_x^- U_{ij}, D_x^+ U_{ij}, D_y^- U_{ij}, D_y^+ U_{ij}) = 0,\ \forall (i,j).$$

The scheme is implicit and we compute the approximation, for given $\Delta x, \Delta y$, with a Gauss-Seidel iteration.

Numerical experiments Figure 1 represents a sphere lying on a base and Figure 2 shows the reconstructed surface from the above original image assuming that the lighting is vertical; finally, Figure 3 represents the intensity corresponding to the reconstructed surface.

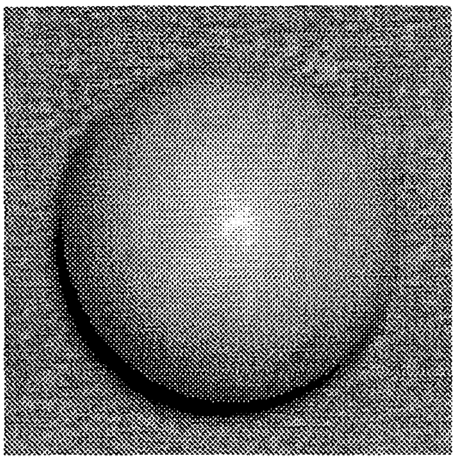

Figure 1: original image

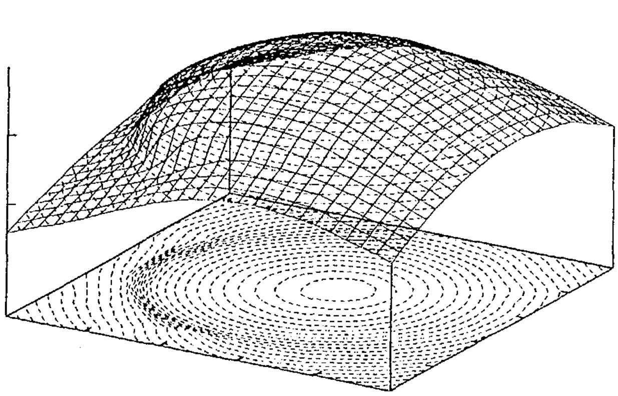

Figure 2: 3D shape

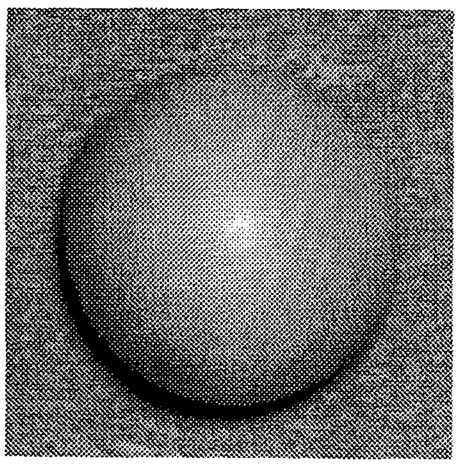

Figure 3: reconstructed image

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