STATISTICAL FROBENIUS MANIFOLDS, LIE GROUPS AND LEARNING

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Abstract. In this paper, n-dimensional statistical manifolds related to exponential families and satisfying the Frobenius manifold axioms are considered. We prove that these manifolds $\mathcal{V}_n$ are equipped with a structure of a Lie group $\mathfrak{G}'$. Since these manifolds can be regarded as a realisation of a module over an algebra generated by a pair of orthogonal idempotents, the submanifolds corresponding to the realisations of the modules over these ideals are totally geodesic. This Lie group $\mathfrak{G}'$ splits into a product of Lie groups $\mathfrak{G} \otimes \mathfrak{G}$, where $\mathfrak{G}$ acts on the respective submanifolds. Moreover, the learning process on Frobenius statistical manifold is equipped with an additional algebraic structure, being the direct product of a continuous transformation group and of a group of affine collineations.

1. Introduction

1.1. The topic of this article is to show important relations between Lie groups and n-dimensional statistical manifolds related to exponential families and satisfying the Frobenius manifold axioms (i.e. Frobenius statistical manifolds). From [7], one can consider Frobenius statistical manifolds from the point of view of paracomplex geometry, i.e. defined as a manifold over the algebra of paracomplex numbers.

The algebra of paracomplex numbers is a rank two (split) algebra generated by 1 and ϵ such that $\epsilon^2 = +1$. This algebra is generated by a pair of (orthogonal) idempotents. Naturally the geometry of the manifold is impacted: from [6] it follows that manifold comes equipped with a pair of submanifolds, corresponding to the realisations of the modules over those ideals.

Our aim here is to show that those Frobenius statistical manifolds are equipped with a structure of a Lie group $\mathfrak{G}'$, and that this Lie group can be splitted itself into a product of Lie groups $\mathfrak{G} \otimes \mathfrak{G}$. Our method is to prove the following statement: if there exists a continuous group of infinitesimal operations on a unital commutative associative algebra then there exists a subgroup isomorphic to it being a Lie group acting on an affine manifold over the above algebra. From the properties of the algebra follows the splitting of $\mathfrak{G}'$.

This furthermore has consequences for the learning process (relying on the Ackley–Hilton–Sejnowski [2]). The learning for the Frobenius statistical manifold is equipped with an additional algebraic structure, being the direct product of a continuous transformation group and of a group of affine collineations.

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1.2. A statistical structure on a differentiable manifold M is a pair (g, C), where g is a metric tensor and C a totally symmetric cubic (0,3)-tensor (i.e., 3-covariant tensor) called the Amari–Chentsov tensor [1, 8]. Such a manifold M can be equipped of the Levi–Civita connection ∇^0 for g, which is the unique torsion free affine connection compatible with the metric g that is satisfying for any triple (X, Y, Z) of vector fields the following equation:

$$X(g(Y,Z)) = g(\nabla^0_X Y, Z) + g(Y, \nabla^0_X Z).$$

Given the Levi–Civita connection ∇^0 for g on M we have a pencil $\{\nabla^\alpha\}$ of α-connections depending on a parameter α defined as:

$$g(\nabla^\alpha_X(Y), Z) := g(\nabla^0_X(Y), Z) - \frac{\alpha}{2} C(X, Y, Z), \tag{1}$$

where X, Y, Z are vector fields in the tangent sheaf T_M. These α-connections being affine and torsion free allow to derive the notion of covariant derivative, parallel transport and geodesics.

Any pair $(\nabla^\alpha, \nabla^{-\alpha})$ defines a pair of conjugate connections $(\nabla, \nabla^\star)$ with respect to the metric g, that is

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^\star_X Z), \tag{2}$$

where $\nabla = \nabla^\alpha$ and $\nabla^\star = \nabla^{-\alpha}$.

The curvature tensor R^α of an α-connection ∇^α is an (1, 3)-tensor given by:

$$g(R^\alpha(X,Y)Z, W) = g([\nabla^\alpha_X, \nabla^\alpha_Y]Z, W) - g(\nabla^\alpha_{[X,Y]} Z, W), \tag{3}$$

where X, Y, Z, W are vector fields on M. In the pencil we have $R^\alpha = R^{-\alpha}$, hence if an α-connection is flat (or locally flat) then, the $-\alpha$-connection is flat as well. For example in the important case of exponential families the α-connections are flat if $\alpha = \pm 1$. This construction provides the possibility of a generalization. Namely, from this construction emerges directly the (pre-)Frobenius manifold structure (i.e. one needs a complete atlas whose transition functions are affine linear, a compatible metric g, an even symmetric rank three tensor and a multiplication operation on the tangent sheaf. Moreover the metric is invariant under the multiplication). The Frobenius structure occurs whenever the associativity and potentiality axioms are given. These axioms imply that the multiplication is associative and that the rank three tensor everywhere, locally admits a potential.

2. Methods

Relying on the fact that the Frobenius manifold is related to an affine manifold over an algebra, it can be seen that the (Lie) group of infinitesimal operators on the given algebra is isomorphic to the subgroup of continuous transformations defined on an affine space over this algebra.

Consider a rank n algebra $\mathfrak{A}_n = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ over the real numbers, where the multiplication is given by $\mathbf{e}_i \cdot \mathbf{e}_j = c^k_{ij}\mathbf{e}_k$. It is a unital algebra equipped with the commutativity and associativity conditions, given respectively by the formulas:

$$c^k_{ij} = c^k_{ji}, \quad c^s_{ik}c^r_{sk} = c^s_{ik}c^r_{sj}. \tag{4}$$

Any element $\mathbf{A}$ of the algebra $\mathfrak{A}_n$ is given by the linear combination

$$\mathbf{A} = A^i \mathbf{e}_i,$$

where A^i is a (real) number and $\mathbf{e}_i$ are generators of $\mathfrak{A}_n$ and the principal unit ϵ is given by $\epsilon^i \mathbf{e}_i$.

In order to show the structure of continuous transformation groups exists on a Frobenius statistical manifold it is sufficient to show the existence of a set of infinitesimal operators on this (unital, commutative, associative, rank n) algebra. In order to do this we need to:

(1) show the existence of parallel transport of vectors on these algebras.
(2) Show the existence of invariant affine connection on the algebra, implying that we can obtain a family of infinitesimal operators of a group of continuous transformations on the algebra.

Concerning (1), we show that the Riemann curvature of the space depends on the type of algebra $\mathfrak{A}_n$. From this statement it follows that the connection depends entirely on the algebra (i.e. on the structure constants of the algebra). Finally, we prove that the affine connection given by the parallel transport of V will be locally Euclidean if and only if the connection Γ is an analytical function of an algebraic function of the variable X.

Furthermore, we prove that the connection Γ—being some geometrical object on the algebra $\mathfrak{A}_n$—is an invariant. To do so, we consider introduce a Killing-like formula for the algebra $\mathfrak{A}_n$.

Let $\xi = \xi(x) = (\xi^1(x), \cdots, \xi^n(x))$ be an analytic functions in the sense of Scheffer. The Killing-like equation for the connection on an algebra $\mathfrak{A}_n$ is given by the following relation:

$$c^s_{ij} c^p_{ks} \partial_j \xi^k + c^b_{jm} c^m_{kb} \partial_i \xi^k = 0, \qquad i, j, \cdots \in \{1, \cdots, n\} \tag{5}$$

where $\partial_i \xi^k$ is a component of a vector field and the c^k_{ij} are structure constants of $\mathfrak{A}_n$.

Finally, we prove that

Theorem 1. *The group of automorphisms of the space $\mathcal{X}_n$ associated to the algebra $\mathfrak{A}_n$ is given by the equation* (5)*.*

Concerning the learning process we refer to the Ackley–Hilton–Sejnowski method, consisting in minimising the Kullback–Leibler divergence. We also use the statistical Gromov–Witten invariants [5] which, similarly to its original version—the (GWS), concern the intersection of (para-)holomorphic curves on the symplectic manifold. This plays an important role in the learning process. In particular we can prove that the (GWS) depend on the structure of the algebra $\mathfrak{A}_n$.

To prove the second result that the learning process on Frobenius statistical manifold is equipped with an additional algebraic structure being the direct product of a continuous transformation group and of a group of affine collineations, we use the following construction.

This is proved using web theory and the construction of Ackley–Hilton–Sejnowski [2]. A d-web of codimension r is given in an open domain D of a differentiable manifold X^{nr} of dimension nr by a set of d foliations of codimension r which are in general position. In fact the geodesics on These geodesics on the submanifolds $\mathcal{V}_p$ (resp. $\mathcal{V}_{n-p}$) allow the construction of the so-called three-webs. Using the first result it implies that the web is a group-web. *In fine*, using the construction of Ackley–Hilton–Sejnowski, we deduce that the learning process depends on two things: the Lie group acting on V_n and the projection collineation group G_{pc} for paracomplex numbers.

3. Results

Theorem A

The n-dimensional statistical manifold $\mathcal{V}_n$ is equipped with a structure of continuous transformations group $\mathfrak{G}'(\mathcal{V}_n)$. The group $\mathfrak{G}'(\mathcal{V}_n)$ is a direct product of continuous transformations groups acting on two sub-manifolds $\mathcal{V}_p$ and $\mathcal{V}_{n-p}$ of the statistical manifold $\mathcal{V}_n$.

Theorem B

The learning process on Frobenius statistical manifold is equipped with an additional algebraic structure being the direct product of continuous transformations groups and a group of affine collineations.

4. Conclusion

We firstly can show the existence of a Lie group on the Frobenius statistical manifolds $\mathcal{V}_n$. We outline the hidden structure of this Lie group on the Frobenius statistical manifold. It can be identified to the direct product of two copies of a given Lie group, each of which act on the totally geodesic submanifolds of $\mathcal{V}_n$, corresponding respectively to the realisations of modules over the orthogonal idempotents of the algebra. The second result is that the learning process for the Frobenius statistical manifold is equipped with an additional algebraic structure being the direct product of the continuous transformation group and of the group of affine collineations.

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