A Linear Complementarity Problem based on non-Symmetric Large Random Matrices

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Résumé – Le problème de complémentarité linéaire (LCP) désigne une classe de problèmes d'optimisation linéaire. Motivés par une question issue de l'écologie théorique, nous étudions les grands systèmes d'équations différentielles couplées de Lotka-Volterra (LV), où l'interaction entre équations est modélisée par une grande matrice aléatoire. Après avoir montré qu'un équilibre stable d'un tel système satisfaisait un problème de LCP, nous analysons ses propriétés statistiques en adaptant des techniques issues de l'*Approximate Message Passing* (AMP), une famille d'algorithmes et techniques développés par Donoho, Montanari et al. ces 15 dernières années. Nous retrouvons ainsi des résultats de Bunin et Galla, établis à l'aide de techniques de physique théorique. Ces concepts de LCP et d'AMP présentent un intérêt pour notre communauté, au-delà de leur utilisation spécifique dans un contexte d'écologie théorique.

Abstract – The Linear Complementarity Problem (LCP) is a class of problems from mathematical optimization. Motivated by a question from theoretical ecology, we study large Lotka-Volterra (LV) systems of coupled differential equations where the interaction between equations is modeled by a large random matrix. After proving that the stable equilibrium of such a system satisfies a LCP, we analyze its statistical properties by adapting techniques from *Approximate Message Passing*, a series of techniques and algorithms developed by Donoho, Montanari et al. these last 15 years. In particular, we recover results established by Bunin and Galla at a physical level of rigor. We believe that these LCP and AMP concepts are of interest to the Statistical Signal Processing community beyond their specific application to a problem of theoretical ecology.

1 Introduction and main result

Large Lotka-Volterra (LV) systems of coupled differential equations, where the coupling between the differential equations is made out of a large non-Symmetric random matrix, are popular in theoretical ecology to model, study and understand food-webs and other large systems in interaction. Of interest is the existence of a stable equilibrium point and its statistical properties: proportion of positive components (representing surviving species) at equilibrium, distribution of these components, etc.

After recalling the formal definition of a Linear Complementarity Problem, we shall prove that the stable equilibrium of a LV system is the solution of a LCP based on the LV system's parameters. Based on recent techniques borrowed from Approximate Message Passing [10, 12], we present a rigorous analysis of the statistical properties of a LCP solution and recover results established by Bunin [4] and Galla [5] at a physical level of rigor.

This work extends to a non-Symmetric setting the results of Akjouj et al. [15].

Linear Complementarity Problem. The Linear Complementarity Problem (LCP) is a class of problems from mathematical optimization which in particular encompasses linear and quadratic programs; standard references are [8, 9]. Given a $n \times n$ matrix M and a $n \times 1$ vector $\boldsymbol{q}$, the associated LCP denoted by $LCP(M, \boldsymbol{q})$ consists in finding two $n \times 1$ vectors $\boldsymbol{z}, \boldsymbol{w}$ satisfying the constraints:

$$\begin{cases} \boldsymbol{z} \geq 0\,, \\ \boldsymbol{w} = M\boldsymbol{z} + \boldsymbol{q} \geq 0\,, \\ \boldsymbol{w}^T \boldsymbol{z} = 0 \quad (\Leftrightarrow \quad w_k z_k = 0\,; \quad 1 \leq k \leq n)\,. \end{cases} \tag{1}$$

Since $\boldsymbol{w}$ can be inferred from $\boldsymbol{z}$, we denote $\boldsymbol{z} \in LCP(M, \boldsymbol{q})$ if $(\boldsymbol{w}, \boldsymbol{z})$ is a solution of (1).

Lotka-Volterra system. Large Lotka-Volterra (LV) systems of differential equations are widely used in various scientific fields involving complex dynamical systems with interacting components, such as biology, ecology, chemistry, etc. [1, 2]. A LV system represents a good trade-off between a fairly realistic model and a mathematically tractable one. In the sequel, we use the ecological terminology and refer to the interacting components as *species*.

A large LV system is a system of differential equations:

$$\frac{dx_k(t)}{dt} = x_k(t) \left(r_k - \theta x_k(t) + \frac{1}{\alpha\sqrt{n}} \sum_{\ell \in [n]} A_{k\ell} x_\ell(t) \right)\,, \tag{2}$$

where $k \in [n] := \{1, \cdots, n\}$.

The number n represents the number of species within the system, the unknown vector $\boldsymbol{x} = (x_k)_{k\in[n]}$ is the $n \times 1$ vector the components of which are solutions to (2) and evolves with time $t > 0$ according to this dynamics. Quantity $x_k(t)$ represents the abundance of species k at time t, a value representing the population size of the species.

In Eq. (2), r_k represents the intrinsic growth rate of species k; we denote by $\boldsymbol{r} = (r_k)$ the $n \times 1$ vector of these rates. Value θ is an intraspecific competition coefficient, and $A_{k\ell}$ is the per capita effect of species ℓ on species k (interactions).

Hereafter, we focus on the idealized model $\theta = 1$:

$$\frac{dx_k}{dt} = x_k \left(r_k - x_k + \frac{(A\boldsymbol{x})_k}{\alpha\sqrt{n}} \right) , \qquad (3)$$

In an ecological or biological context (think of animal species interacting in a lake or a remote valley, or the human microbiome), it is often extremely difficult and/or expensive to estimate precisely each interaction strength $A_{k\ell}$. In the absence of any prior information, these interactions can be modeled as random (see for instance [3]), which we assume in the sequel:

Assumption 1. *matrix $(A_{k\ell})_{k,\ell \in [n]}$ is a $n \times n$ matrix of independent and identically distributed (i.i.d.) standard Gaussian $\mathcal{N}(0,1)$ random variables (RV).*

Notice that each variable $A_{k\ell}$ is multiplied by the normalizing factor $(\alpha\sqrt{n})^{-1}$. The positive number α is an extra parameter reflecting the interaction strength.

The equilibrium of a LV system satisfies a LCP. A key element to understand the dynamics of (2) is the existence of an equilibrium $\boldsymbol{x}^* = (x_k^*)_{k\in[n]}$ such that

$$x_k^* \left(r_k - x_k^* + \frac{(A\boldsymbol{x}^*)_k}{\alpha\sqrt{n}} \right) = 0 \quad \forall k \in [n] , \qquad (4)$$

and the study of its stability, that is the convergence of a solution $\boldsymbol{x}$ to the equilibrium $\boldsymbol{x}^*$: $\boldsymbol{x}(t) \xrightarrow[t\to\infty]{} \boldsymbol{x}^*$ if $\boldsymbol{x}(0)$ is sufficiently close to $\boldsymbol{x}^*$.

It is well known that for LV equations, the fact that $\boldsymbol{x}(0) > 0$ (componentwise) implies that $\boldsymbol{x}(t) > 0$ for every $t > 0$, but one can have some components $x_k(t)$ of $\boldsymbol{x}(t)$ vanishing to zero. We hence only consider non-negative equilibria $\boldsymbol{x}^* \geq 0$.

Relying on standard properties of dynamical systems, see for instance [7, Theorem 3.2.5], a necessary condition for the equilibrium $\boldsymbol{x}^*$ to be stable is that

$$r_k - x_k^* + \frac{(A\boldsymbol{x}^*)_k}{\alpha\sqrt{n}} \leq 0 . \qquad (5)$$

As we shall see, this casts the problem of finding a non negative equilibrium into the class of LCP.

Denote $\check{A} = \frac{A}{\alpha\sqrt{n}}$. Gathering the constraints of the equilibrium $\boldsymbol{x}^*$ defined in (4) and (5), we get:

$$\begin{cases} \boldsymbol{x}^* \geq 0 , \\ r_k - x_k^* + (\check{A}\boldsymbol{x}^*)_k \leq 0 , \\ x_k \left(r_k - x_k^* + (\check{A}\boldsymbol{x}^*)_k \right) = 0 . \end{cases}$$

Otherwise stated, $\boldsymbol{x}^* \in LCP(I - \check{A}, -\boldsymbol{r})$.

Based on Takeuchi and Adachi's theorem [7], we have proved in [13] (see also [14]) that $LCP(I-\check{A}, -\boldsymbol{r})$ eventually admits a unique solution if $\alpha > \sqrt{2}$ and that this equilibrium is globally stable for the system (2), for any initial condition $\boldsymbol{x}(0) > 0$.

An interesting and highly non-trivial question is the following: is it possible (and if so, how?) to extract statistical information for $\boldsymbol{x}^*$ from the random matrix model of $\check{A}$? As we shall see, the answer is positive and relies on Approximate Message Passing techniques.

Main result: statistical properties of the LCP solution. In the sequel, we assume that $\alpha > \sqrt{2}$. This implies that for a given realization ω, there exists $N(\omega)$ such that $LCP(I - \check{A}_\omega, -\boldsymbol{r})$ admits a unique solution $\boldsymbol{x}^* = (x_i^*)_{i\in[n]}$ for any $n \geq N(\omega)$. Denote by $\xrightarrow{W_2}$ the 2-Wasserstein convergence of probability measures, that is the convergence for every test function f continuous and sub-quadratic, i.e. $|f(x)| \leq K(1+|x|^2)$. We also assume that there exists a nonnegative real random variable $\underline{r}$ satisfying $\mathbb{P}(\underline{r} > 0) > 0$ such that

$$\mu^{\boldsymbol{r}} = \frac{1}{n} \sum_{i\in[n]} \delta_{r_i} \xrightarrow[n\to\infty]{W_2} \mathcal{L}(\underline{r}) . \qquad (6)$$

We finally assume that $\boldsymbol{r}$ is independent from A.

We are now in position to state the main theorem:

Theorem 1. *Let $\alpha > \sqrt{2}$. Denote by $\mu^{\boldsymbol{x}^*}$ the empirical measure*

$$\mu^{\boldsymbol{x}^*} = \frac{1}{n} \sum_{i\in[n]} \delta_{x_i^*} .$$

Let $Z \sim \mathcal{N}(0,1)$ and consider the following fixed point equation with unknown $\sigma > 0$:

$$\sigma^2 = \frac{1}{\alpha^2} \mathbb{E}(\sigma Z + \underline{r})_+^2 \quad \textit{where} \quad x_+ = \max(x, 0) . \qquad (7)$$

Then this equation admits a unique solution $\sigma > 0$ and there exists a random variable $Y \stackrel{\mathcal{D}}{=} (\sigma Z + \underline{r})_+$ such that

$$\mu^{\boldsymbol{x}^*} \xrightarrow[n\to\infty]{W_2} \mathscr{L}(Y) \quad \textit{a.s.}$$

Otherwise stated, for every test function f continuous and subquadratic, a.s.

$$\frac{1}{n} \sum_{i\in[n]} f(x_i^*) \xrightarrow[n\to\infty]{} \mathbb{E} f(Y) .$$

Remark 1.
1. *The set of test functions contains the continuous bounded functions, so that (a.s.) $\mu^{\boldsymbol{x}^*}$ converges weakly to the distribution of Y.*
2. *As will be illustrated by the simulations, (see Fig 1), the quantity $\mathbb{P}\{\sigma Z + \underline{r} > 0\}$ is a good proxy for the empirical proportion of surviving species*

$$\frac{\#\{x_i^* > 0\}}{n} = \frac{1}{n} \sum_{i\in[n]} 1_{[0,\infty)}(x_i^*)$$

(although the theorem does not provide a theoretical guarantee since the function $x \mapsto 1_{[0,\infty)}(x)$ is not continuous).

3. *Theorem 1 provides information on the distribution of surviving species. On the one hand, one can easily prove that the distribution $\mathcal{L}(Y \mid Y > 0)$ admits the density:*

$$f_{Y|Y>0}(y) = \frac{1_{(y>0)}}{\mathbb{P}(\sigma Z + \underline{r} > 0)} \int \frac{e^{-\frac{(y-r)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \mathbb{P}_{\underline{r}}(dr) \,. \tag{8}$$

On the other hand, one can plot the histogram of the positive components $x_i^* > 0$ of $\boldsymbol{x}^*$. Simulations illustrate a good matching between the theoretical distribution and its empirical counterpart, see Figure 2.

2 Elements of proof

A fixed point equation associated to the LCP. Let $\boldsymbol{y} = (y^i)_{i\in[n]} \in \mathbb{R}^n$ and denote by $\boldsymbol{y}_\pm = (y^i_\pm)_{i\in[n]}$, where $x_+ = \max(x,0)$ and $x_- = \max(-x,0)$. Notice that $\boldsymbol{y} = \boldsymbol{y}_+ - \boldsymbol{y}_-$ and that $y^i_+ \cdot y^i_- = 0$ for $i \in [n]$.

Proposition 2. *Let $\boldsymbol{y} \in \mathbb{R}^n$ the solution of the fixed-point equation*

$$\boldsymbol{y} = \check{A}\boldsymbol{y}_+ + \boldsymbol{r} \,. \tag{9}$$

Then $\boldsymbol{y}_+ \in LCP(I - \check{A}, -\boldsymbol{r})$.

Reciprocally, let $\boldsymbol{x}^* \in LCP(I - \check{A}, -\boldsymbol{r})$. Denote by

$$\boldsymbol{y}_- = (I - \check{A})\boldsymbol{x}^* - \boldsymbol{r} \geq 0 \,,$$

then $\boldsymbol{y} = \boldsymbol{x}^* - \boldsymbol{y}_-$ is a solution to Eq. (9)

Proof. Suppose that $\boldsymbol{y}$ satisfies (9). Replacing $\boldsymbol{y}$ by $\boldsymbol{y}_+ - \boldsymbol{y}_-$ in (9), we obtain $(I - \check{A})\boldsymbol{y}_+ - \boldsymbol{r} = \boldsymbol{y}_- \geq 0$. Since $\boldsymbol{y}_+ \geq 0$ and $y_i^+ \cdot y_i^- = 0$, we have $\boldsymbol{y}_- \in LCP(I - \check{A}, -\boldsymbol{r})$. The converse is immediate.

□

An iterative scheme to build the LCP solution. Consider the following iterative scheme

$$\begin{cases} \boldsymbol{z}^0 = \boldsymbol{0}, \\ \boldsymbol{z}^{p+1} = \check{A}\left(\boldsymbol{z}^p + \boldsymbol{r}\right)_+ = \frac{A}{\sqrt{n}} \frac{(\boldsymbol{z}^p + \boldsymbol{r})_+}{\alpha} \,. \end{cases} \tag{10}$$

The following result can be established by induction.

Theorem 3. *Let $p \in \mathbb{N}$ be fixed and let $\boldsymbol{z}^1, \cdots, \boldsymbol{z}^p$ be defined by* (10)*. Recall that $\boldsymbol{r}$ satisfies* (6) *and is independent from A. Then*

$$\mu^{(\boldsymbol{z}^1,\cdots,\boldsymbol{z}^p)} = \frac{1}{n}\sum_{i\in[n]} \delta_{(z_i^1,\ldots,z_i^p)} \xrightarrow[n\to\infty]{W_2} (Z^1,\cdots,Z^p) \text{ a.s.},$$

where $(Z^1,\cdots,Z^p)$ is a centered Gaussian vector whose covariance matrix is given by

$$\mathbb{E}Z^iZ^j = \frac{1}{\alpha^2}\mathbb{E}(Z^{i-1} + \underline{r})_+(Z^{j-1} + \underline{r})_+ \quad \text{for} \quad i,j \in [n] \,,$$

where by convention, $Z^0 = 0$. In particular, if we denote by $\sigma_p^2 = \mathrm{var}(Z^p)$, then

$$\sigma_{p+1}^2 = \frac{1}{\alpha^2}\mathbb{E}(\sigma_p Z + \underline{r})_+^2 \quad \text{where } Z \sim \mathcal{N}(0,1) \,.$$

Theorem 3 is the main building block of proof of Theorem 1. It can be proved by an induction over $p \geq 1$ which is strongly inspired by AMP-type results. By a standard argument, we can prove that $\sigma_p \xrightarrow[p\to\infty]{} \sigma$, where σ is solution of (7).

Another important argument lies in the fact that the Z^p's become more and more correlated as $p \to \infty$, which is proved by establishing that

$$\frac{\mathbb{E}Z^pZ^{p-1}}{\sigma_p\sigma_{p-1}} \nearrow 1 \quad \text{as} \quad p \to \infty \,.$$

This argument is borrowed from Montanari and Richard [11] and results in the fact that vectors $\boldsymbol{z}^p$ and $\boldsymbol{z}^{p+1}$ tend to be aligned for large p, after $n \to \infty$. Setting $\boldsymbol{y}^p = \boldsymbol{z}^p + \boldsymbol{r}$ and approximating $\boldsymbol{y}^{p+1}$ by $\boldsymbol{y}^p$, Eq. (10) writes

$$\boldsymbol{y}^p = \check{A}\boldsymbol{y}^p_+ + \boldsymbol{r} + \boldsymbol{\varepsilon}^p \,,$$

where $\boldsymbol{\varepsilon}^p$ accounts for the approximation $\boldsymbol{y}^{p+1} \simeq \boldsymbol{y}^p$. This last equation is an approximated version of (9) and reads

$$\boldsymbol{y}^p \in LCP(I - \check{A}, -\boldsymbol{r} - \boldsymbol{\varepsilon}^p) \,.$$

The last argument to establish Theorem 1 is a perturbation result for the LCP [16] which roughly states that

$$\boldsymbol{t}^\varepsilon \in LCP(M^\varepsilon, \boldsymbol{q}^\varepsilon) \to \boldsymbol{t} \in LCP(M, \boldsymbol{q})$$

as $(M^\varepsilon, \boldsymbol{q}^\varepsilon) \xrightarrow[\varepsilon\to 0]{} (M, \boldsymbol{q})$ as long as the solutions are unique.

3 Simulations

In this section, we provide simulations where the theoretical and asymptotic results provided in the theorems are compared to their finite and empirical counterparts. As it appears, the matching is very good.

4 Conclusion

In this article, we have studied the stable equilibrium point of the Lotka-Volterra system (2) in the case where the $n \times n$ interaction matrix A has random $\mathcal{N}(0,1)$ i.i.d. entries. If $\alpha > \sqrt{2}$ then there eventually exists a unique equilibrium $\boldsymbol{x}_n^\star$ which is itself random, due to the randomness of A, and satisfies the LCP $LCP(I - \check{A}, -\boldsymbol{r})$ where $\check{A} = \frac{A}{\alpha\sqrt{n}}$. Considering the associated empirical measure $\mu^{\boldsymbol{x}^\star} = \frac{1}{n}\sum_{i\in[n]} \delta_{x_i^\star}$, we establish that $\mu^{\boldsymbol{x}^\star}$ converges towards the distribution of $(\sigma Z + \underline{\mathrm{r}})_+$ where $Z \sim \mathcal{N}(0,1)$, σ satisfies the following fixed-point equation

$$\sigma^2 = \frac{1}{\alpha^2}\mathbb{E}(\sigma Z + \underline{\mathrm{r}})_+^2$$

and $\underline{\mathrm{r}}$ is a random variable independent from Z whose distribution is the limit of $\mu^{\boldsymbol{r}}$.

Our result is based on a novel Approximate Message Passing type algorithm designed to handle the non-Symmetric matrix A. We believe that the interest of this result and method goes beyond its mere application to theoretical ecology problems.

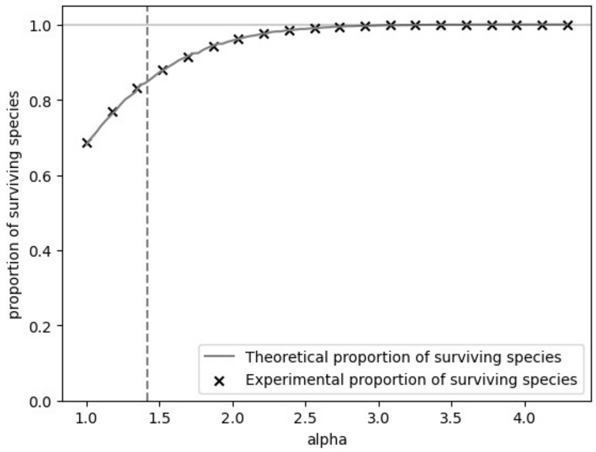

Figure 1: The plot represents a comparison between the theoretical proportion of surviving species (continuous line) and its empirical counterpart - see Remark 1-(2). Size n is set to 10000; parameter α on the x-axis ranges from 1 to $\sqrt{2\log(n)} \simeq 4.29$. The threshold $\alpha > \sqrt{2}$ (vertical dotted line) represents the theoretical guarantee to have a stable equilibrium; $\alpha = \sqrt{2\log(n)}$ is the upper-limit above which we have no extinction ($p^* = 1$). Notice that for $\alpha \in [1, \sqrt{2}]$, the heuristics shows a remarkable matching with the empirical data despite no theoretical guarantees.

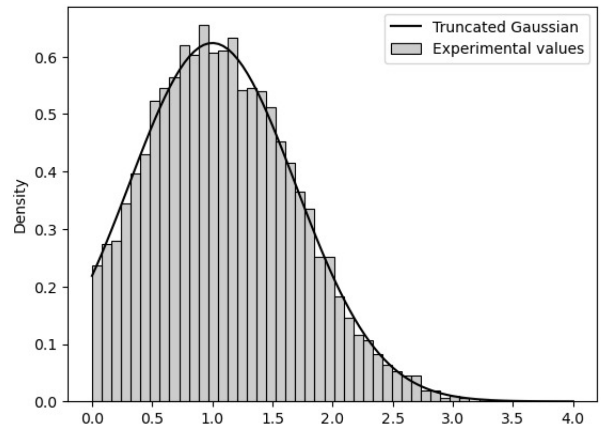

Figure 2: The continuous line represents the density (8) in the case where $\underline{\mathrm{r}} = \mathbf{1}$, that is the density of a truncated Gaussian. The histogram is based on the positive components of the equilibrium $\boldsymbol{x}$.

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