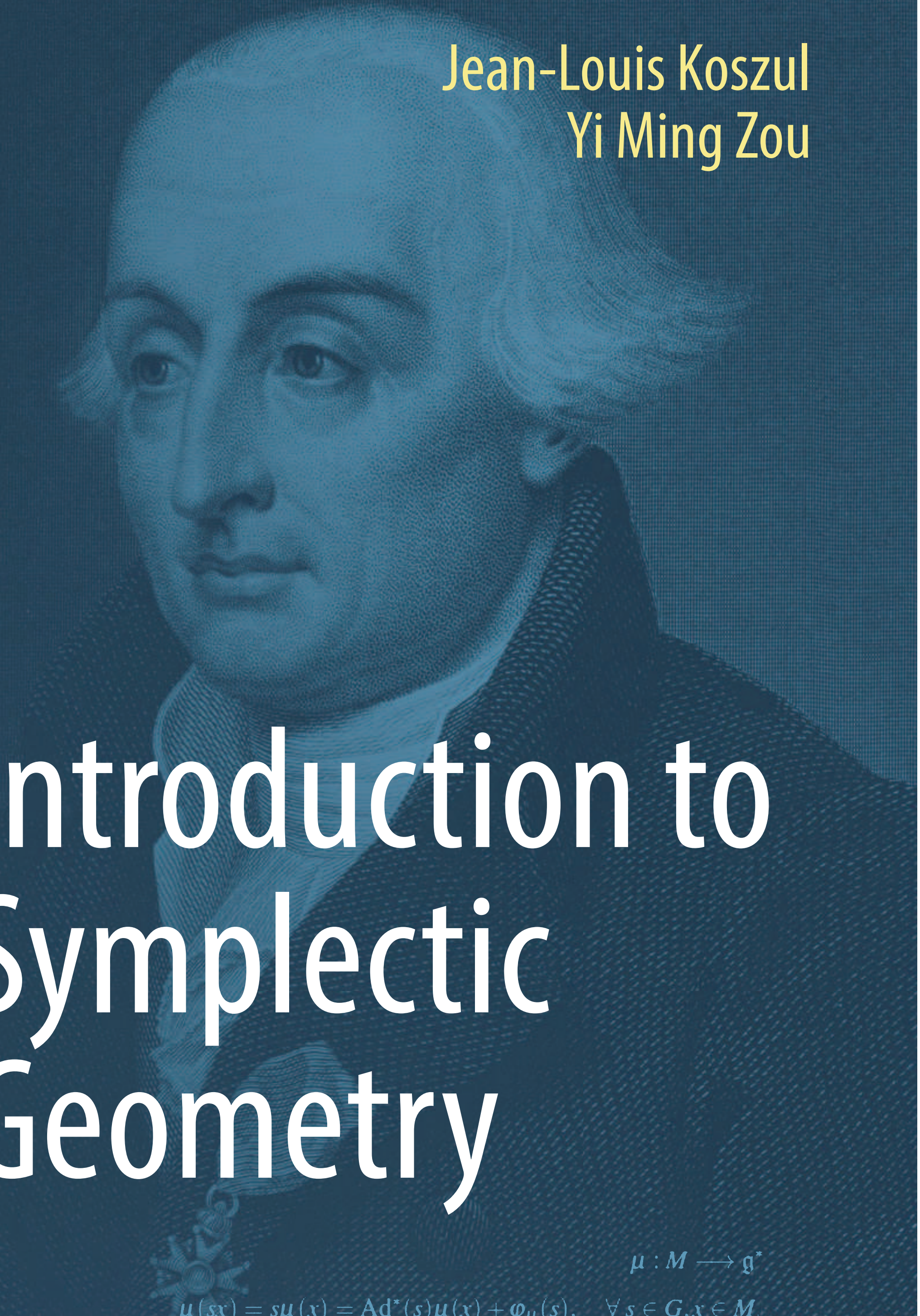
Jean-Louis Koszul
Yi Ming Zou



Introduction to Symplectic Geometry

$$\mu : M \longrightarrow \mathfrak{g}^*$$

$$\mu(sx) = s\mu(x) = \mathrm{Ad}^*(s)\mu(x) + \varphi_\mu(s), \quad \forall\, s \in G, x \in M$$

$$c_\mu(a,b) = \{\langle \mu, a\rangle, \langle \mu, b\rangle\} - \langle \mu, [a,b]\rangle = \langle \mathrm{d}\,\varphi_\mu(a), b\rangle, \quad \forall\, a,b \in \mathfrak{g}$$

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Foreword 1. About This Book

En effet, Barbaresco m'a dit qu'il était curieux de voir le texte qui a été publié d'après un cours sur la géométrie symplectique que j'ai donné à Tianjin, il y a bien longtemps. Je n'ai rien pu lui procurer d'autre que le petit fascicule rédigé en chinois par un assistant de Nankai. Je ne sais pas ce qu'il vaut. De toutes façons, il n'y avait pas beaucoup de choses originales dans ce que j'ai raconté.

Indeed, Barbaresco told me that he was curious to see the text that was published according to a course on symplectic geometry that I gave in Tianjin, long time ago. I could not give anything else but the little notebook written in Chinese by a Nankai assistant. I do not know whether it was worthful. In any case, there were no too many original things in what I was speaking about.

J.-L. Koszul, 02/02/2017.

Above is an excerpt from a message sent to me by Jean-Louis Koszul in February 2017. Earlier, in January 2017, I had written to Koszul about the lectures on Symplectic Geometry he delivered in China. I informed him that the notes of those lectures were missing in my private documentation. I also informed him that Fréderic Barbaresco was interested in these notes.

This book is more than an elementary introduction to symplectic structures and their geometry. It highlights the unifying nature of symplectic structures. Often, readings of Koszul works are walks through *Algebra, Homological Algebra, Geometry, Differential Geometry, Topology*, and *Differential Topology*. This adage is highlighted by Foreword 3. The lectures in this book were delivered while new developments in Symplectic Geometry were occurring, with the works of A. Weinstein, B. Kostant, V. Guillemin, S. Stenberg, M. Atiyah, R. Bott, and many others. Koszul also emphasizes fruitful exchanges of results and techniques between Symplectic Geometry and other areas. For example, in Chap. 4, the notion of momentum map leads to rich relationships between symplectic structures, homological algebra, affine representations of Lie groups and Lie algebras, and homogeneous spaces. The author introduces many additional structures in manifolds endowed with symplectic structures. Major instances are Lagrangian submanifolds, complex symplectic structures, and Kaehler forms. Symplectic structures appear as a unifying framework for several notions worth studying. Koszul's Geometry and its applications to bounded domains strongly impact many areas which are currently the subject of active and exciting research, such as the Geometric Science of Information and the Topology of Information. According to the Stefan–Sussmann theorem, Poisson geometry is the Differential Topological side of symplectic geometry, while Lagrangian foliations are the differential topological side of locally flat geometry. The starting point of this book is the algebraic counterpart of those subjects. One could claim that quantum data are those data which are $\mathbb{Z}_2$-graded. Under such a simplification, Chap. 6 is an introduction to quantum symplectic structures. I have often talked with Koszul about his stays in China and in India. He and the editors had planned to include a video with this book. However, on November 28, 2017, Koszul wrote to me :

Depuis que nous sommes installés dans cette maison de retraite, je suis très mal en point et très affaibli. Dans l'état où je suis, il n'est évidemment pas question que je donne un interview, je regrette bien de vous décevoir.

J.-L. Koszul, 28/11/2017.

Since we have settled in this retirement house, I feel very badly and very weakened. In the situation that I am now, there is obviously no question that I give an interview, I regret to disappoint you.

What is new? As I just mentioned, some of Koszul's work deeply impacts current research in both the Topology of Information and the Geometric Science of Information as well as their applications in Physics. This relates to his work on the geometry of convex cones, on the affine representations of Lie groups and Lie algebras, and on deformations of locally flat manifolds. Koszul knew of the connection through Barbaresco's paper *Koszul information geometry and Souriau Lie group thermodynamics*. He wrote to me :

Pour ce qui est des représentations affines, je ne suis pas le premier à les avoir manipulé. Si j'ai bon souvenir, elles interviennent dans le travail des russes sur les domaines bornés. A part cela, cet article de Barbaresco contient bien de choses que j'aimerais comprendre. Je vais essayer de m'y mettre.

As for the affine representations, I am not the first to have manipulated them. If I remember correctly, they intervene in the work of the Russians on the bounded domains. Apart from that, this Barbaresco article contains many things that I would like to understand. I will try myself to enter into the matter of the article.

J.-L. Koszul.

Thence, I undertook to convince him that the mathematical foundation of the Geometric Science of Information is the algebraic topology of Koszul Geometry, viz the cohomology theory of Koszul Geometry. Subsequently, Koszul's interest in the Geometric Science of Information increased.

Je suis sensible à l'honneur que me fait le comité d'organisation du congrès 2013 en m'invitant à la conférence de Shima et je vous demande de bien vouloir lui transmettre mes remerciements. J'aimerais aussi pouvoir vous dire de transmettre mon acception.

I am sensible to the honor that the organizing committee of the 2013 congress gave me inviting me to the Shima conference and I ask you to forward kindly my thanks. I would also like to tell you to convey my gratitude.

J.-L. Koszul, 20/12/2012.

On August 29, 2013, H. Shima delivered a keynote conference on Hessian Geometry, whose founder is Jean-Louis Koszul.

Je ne regrette pas d'avoir été à Paris le 29 Août, en plus de ces retrouvailles avec Shima, j'ai observé avec intérêt ce colloque GSI dont le contenu et les objectifs étaient pour moi assez mystérieux. J'ai aussi regardé avec curiosité le volume de Lectures Notes publié à l'occasion de cette rencontre et cela m'a bien aidé à comprendre ce que l'on visait. A propos de ce volume, réussir à le sortir dans les délais est une prouesse que j'admire beaucoup. Je crois bien n'avoir jamais vu cela. Encore une fois, merci de m'avoir signalé cette rencontre et de m'avoir encouragé à faire le déplacement.

I do not regret having been in Paris on August 29, in addition to this reunion of Shima, I watched with interest this GSI conference whose content and objectives were rather mysterious for me. I also looked with curiosity at the volume of Lecture Notes published on the occasion of this meeting and it helped me to understand what we sought. Concerning this volume, succeeding to release on time is a feat that I admire a lot. I think I have never seen things like that. Once again, thank you for informing me about this meeting and to have encouraged me to make the trip.

J.-L. Koszul.

Montpellier, France

Michel Nguiffo Boyom Emeritus Professor
IMAG: Alexander Grothendieck Research Institute
University of Montpellier

Foreword 2. Koszul Contemporaneous Lecture: Elementary Structures of Information Geometry and Geometric Heat Theory

La Physique mathématique, en incorporant à sa base la notion de groupe, marque la suprématie rationnelle… Chaque géométrie—et sans doute plus généralement chaque organisation mathématique de l'expérience—est caractérisée par un groupe spécial de transformations…. Le groupe apporte la preuve d'une mathématique fermée sur elle-même. Sa découverte clôt l'ère des conventions, plus ou moins indépendantes, plus ou moins cohérentes

Gaston Bachelard, Le nouvel esprit scientifique, 1934

Jean-Louis Koszul's Life: The Spirit of Geometry and the Spirit of Finesse of an "Esprit raffiné"

Jean-Louis André Stanislas Koszul, born in Strasbourg in 1921, was the fourth child of a family of four (with three older sisters, Marie Andrée, Antoinette, and Jeanne) of André Koszul (born in Roubaix on November 19, 1878, Professor at Strasbourg University), and Marie Fontaine (born in Lyon on June 19, 1887), who was a friend of Henri Cartan's mother. Henri Cartan writes on this friendship "*My mother, in her youth, had been a close friend of the one who was to become Jean-Louis Koszul's mother*" [4]. His paternal grandparents were Julien Stanislas Koszul and Hélène Ludivine Rosalie Marie Salomé. He attended high school in Fustel-de-Coulanges in Strasbourg and the Faculty of Science in Strasbourg and in Paris. He entered ENS Ulm in the class of 1940 and defended his thesis with Henri Cartan. Henri Cartan noted "*This promotion included other mathematicians like Belgodère and Godement, and also physicists and some chemists, like Marc Julia and Raimond Castaing*" [4] (Fig. 1).

On July 17, 1948, Jean-Louis Koszul married Denise Reyss-Brion, who became a student at ENS Sèvre, in 1941. They had three children, Michel (married to Christine Duchemin), Anne (wife of Stanislas Crouzier), and Bertrand. Koszul then taught in Strasbourg and was appointed as Associate Professor at the University of Strasbourg in 1949, and his colleagues including René Thom, Marcel Berger, and Bernard Malgrange. He was promoted to Professor in 1956 and became a member of second generation of Bourbaki, with Jacques Dixmier, Roger Godement, Samuel

Fig. 1 ENS Ulm students 1940

Eilenberg, Pierre Samuel, Jean-Pierre Serre, and Laurent Schwartz. Henri Cartan remarked in [4] "*In the vehement discussions within Bourbaki, Koszul was not one of those who spoke loudly; but we learned to listen to him because we knew that if he opened his mouth he had something to say*" (Fig. 2).

About Koszul's period at Strasbourg University, Pierre Cartier [43] said "*When I arrived in Strasbourg, Koszul was returning from a year spent at the Institute for*

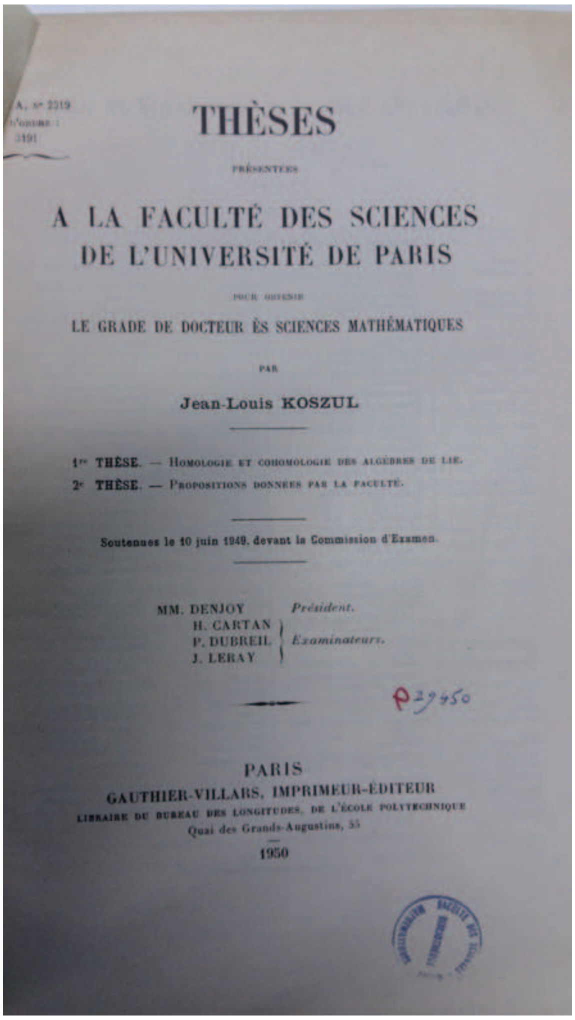
Fig. 2 Jean-Louis Koszul at the Bourbaki seminar 1951

Fig. 3 Jean-Louis Koszul at the differential topology colloquium, Strasbourg 1953; second row from the bottom, Koszul is the second person from the left before André Weil. We can also see in the picture Chern, de Rham, Eckmann, Ehresmann, Godeaux, Hopf, Lichnerowicz, Malgrange, Milnor, Reeb, Schwartz, Süss, Thom, and Libermann

Advanced Studies in Princeton, and he was, after the departure of Ehresman and Lichnerowicz to Paris, the paternal figure of the Department of Mathematics (despite his young age). I am not sure of his intimate convictions, but he represented for me a typical figure of this Alsatian Protestantism, which I frequented at the time. He shared the seriousness, the honesty, the common sense and the balance. In particular, he knew how to resist the academic attraction of Paris. He left us after two years to go to Grenoble, in a maneuver uncommon at the time, exchanging of positions with Georges Reeb." In Strasbourg, he supervised Edith Kosmanek Ph.D. [44], a graduate from Louis Pasteur University (Fig. 3).

Koszul became Senior Lecturer at the University of Grenoble in 1963, and then an Honorary Professor at Joseph Fourier University [6] and integrated into the Fourier Institute led by Claude Chabauty. During this period, as recalled by Bernard Malgrange [42], Koszul held a seminar on "*algebra and geometry*" with his three students Jacques Vey [45–46], Domingo Dominique Luna [47], and Jacques Helmstetter [48–50]. In Grenoble, Koszul practiced mountaineering and was a member of the French Alpine Club. He was awarded the Jaffré Prize in 1975 and was elected correspondent at the Academy of Sciences on January 28, 1980. The following year, he was elected to the Academy of São Paulo. Koszul was one of the CIRM conference center founders at Luminy. Jean-Louis Koszul died on January 12, 2018, at the age of 97.

As early as 1947, Koszul published three articles in the Comptes Rendus of the Academy of Sciences, on the Betti number of a simple compact Lie group, on cohomology rings, generalizing ideas of Jean Leray, and finally on the homology of homogeneous spaces. Koszul's thesis, defended on June 10, 1949 under the direction of Henri Cartan, dealt with the homology and cohomology of Lie algebras. The jury was composed of Professors Arnaud Denjoy (president), Henri Cartan, Paul Dubreil, and Jean Leray. Under the title "*Works of Koszul I, II and III*", Henri Cartan reported Koszul's Ph.D. results to the Bourbaki seminar (Fig. 4).



THÈSES

PRÉSENTÉES

A LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ DE PARIS

POUR OBTENIR

LE GRADE DE DOCTEUR ÈS SCIENCES MATHÉMATIQUES

PAR

Jean-Louis KOSZUL

1re THÈSE. — Homologie et cohomologie des algèbres de Lie.
2e THÈSE. — Propositions données par la Faculté.

Soutenues le 10 juin 1949, devant la Commission d'Examen.

MM. DENJOY Président.
H. CARTAN
P. DUBREIL Examinateurs.
J. LERAY

PARIS
GAUTHIER-VILLARS, IMPRIMEUR-ÉDITEUR
LIBRAIRE DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE
Quai des Grands-Augustins, 55

1950

Fig. 4 Cover page of Koszul's Ph.D. report, defended on June 10, 1949 with a Jury composed of Profs. Arnaud Denjoy, Henri Cartan, Paul Dubreil, and Jean Leray

In 1987, an International Symposium on Geometry was held in Grenoble in honor of Koszul, whose proceedings were published in "les Annales de l'Institut Fourier", Volume 37, No. 4. This conference began with a presentation by Henri Cartan, who remembered the mention given to Koszul for his aggregation [4]: "*Distinguished Spirit; he is successful in his problems. Should beware, orally, of overly systematic trends. A little less subtle complications, baroque ideas, a little more common sense and balance would be desirable.*" About his supervision of Koszul's Ph.D., Henri Cartan wrote "*Why did he turn to me as his 'guide' (so to speak)? Is it because he found inspiration in Elie Cartan's work on the topology of Lie groups? Perhaps he was surprised to note that mathematical knowledge is not necessarily transmitted by descent. In any case, he helped me to better know what my father had brought to the theory*" [4]. On the work of Koszul algebraic work, Henri Cartan notes "*Koszul was the first to give a precise algebraic formalization of the situation studied by Leray in his 1946 publication, which became the theory of spectral sequences. It took a good deal of insight to unravel what lay behind Leray's study. In this respect, Koszul's Note in the July 1947 CRAS is of historical significance*" [4]. From June 26 to July 2, 1947, CNRS hosted an International conference in Paris, on "*Algebraic Topology*". This was the first postwar international diffusion of Leray's ideas. Koszul writes about this lecture "*I can still see Leray putting down his chalk at the end of his talk by saying (modestly?) that he definitely did not understand anything about Algebraic Topology*." In writing his lectures at the Collège de France, Leray adopted the algebraic presentation of the spectral sequence elaborated by Koszul. As early as 1950, Jean-Pierre Serre used the term "*Leray–Koszul sequence*". Speaking of Leray, Koszul wrote "*around 1955, I remember asking him what had put him on the path of what he called the homology ring of a representation in his Notes to the CRAS of 1946. His answer was Künneth's theorem; I could not find out more*." Sheaf theory, introduced by Jean Leray, followed in 1947, at the same time as spectral sequences.

In 1950, Koszul published an important 62-page book entitled *Homology and Cohomology of Lie Algebras* in which he studied the links between the homology and cohomology (with real coefficients) of a compact connected Lie group and purely algebraic problems of Lie algebra. Koszul then gave a lecture in São Paulo on the topic "*sheaves and cohomology*". The superb lecture notes were published in 1957 and dealt with Čech cohomology with coefficients in a sheaf. In the autumn of 1958, he organized a second series of seminars in São Paulo, on symmetric spaces [8]. R. Bott commented on these seminars "*very pleasant. The pace is fast, and the considerable material is covered elegantly. In addition to the more or less standard theorems on symmetric spaces, the author discusses the geometry of geodesics, Bergmann's metrics, and finally studies the bounded domains with many details*." In the mid-1960s, Koszul taught at the Tata Institute in Bombay on transformation groups [12] and on fiber bundles and differential geometry. The second lecture dealt with the theory of connections and the lecture notes were published in 1965.

Fig. 5 Henri Cartan lecture on homogeneous domains, Freiburg, March 13, 1987

In 1994, in [3], a comment by Koszul explains the problems he was preoccupied with when he invented what is now called the "*Koszul complex*". This was introduced to define a cohomology theory for Lie algebras and proved to be useful in general homological algebra.

Fig. 6 (On the left) Yann Ollivier and Jean-Louis Koszul in the GSI'13 conference group photo at Hôtel de Vendôme, Ecole des Mines de Paris, (on the right) Jean-Louis Koszul at the CIRM anniversary in Luminy

The Genesis of this Translation of Koszul's Book "*Introduction to Symplectic Geometry*"

The genesis of the translation of this book dates back to 2013. We got in contact with Professor Koszul in connection with his work on homogeneous bounded domains and their links with Information Geometry. Professor Michel Boyom successfully convinced Professor Koszul to accept our invitation to attend the first GSI "*Geometric Science of Information*" conference in August 2013 at Ecole des Mines ParisTech in Paris, and more especially to attend Hirohiko Shima talk, given in his honor, on the topic "*Geometry of Hessian Structures*" (Fig. 7).

We were more particularly motivated by Koszul's work developed in his paper ***Domaines bornées homogènes et orbites de groupes de transformations affines*** [9] of 1961, written by Koszul at the Institute for Advanced Studies in Princeton during a stay funded by the National Science Foundation. Koszul proved in this paper that on a complex homogeneous space, an invariant volume defines with the complex structure the canonical invariant Hermitian form introduced in [7]. It is in this article that Koszul uses the ***affine representation of Lie groups and Lie algebras***. By studying the open orbits of the affine representations, he introduced an affine representation of G, written$(\mathbf{f}, \mathbf{q})$, and the following equation setting f of the

Fig. 7 Jean-Louis Koszul and Hirihiko Shima at the GSI'13 conference in Ecole des Mines ParisTech in Paris, October 2013

linear representation of the Lie algebra $\mathfrak{g}$ of G, defined by $\mathbf{f}$, and q the restriction to $\mathfrak{g}$ of the differential of $\mathbf{q}$ (f and q are, respectively, differential of $\mathbf{f}$ and $\mathbf{q}$):

$$\begin{gathered} f(X)q(Y) - f(Y)q(X) = q([X, Y])\ \forall X, Y \in \mathfrak{g} \\ \text{with}\ f : \mathfrak{g} \to gl(E)\ \text{ and }\ q : \mathfrak{g} \mapsto E \end{gathered} \tag{1}$$

If the homogeneous space is holomorphically isomorphic to a bounded domain of a space C^n, this Hermitian form is positive definite because it coincides with the Bergmann metric of the domain. Koszul demonstrates in this article the converse of this proposition for a class of complex homogeneous spaces. This class consists of some open orbits of complex affine transformation groups and contains all homogeneous bounded domains. Koszul again addresses the problem of knowing if a complex homogeneous space, with a canonical Hermitian form, that is positive definite, is isomorphic to a bounded domain, but via the study of the invariant bilinear form defined on a real homogeneous space by an invariant volume and an invariant flat connection. Koszul demonstrates that if this bilinear form is positive definite, then the homogeneous space with its flat connection is isomorphic to a convex open domain containing no straight line in a real vector space and extends it to the initial problem for the complex homogeneous spaces obtained in defining a complex structure in the variety of vectors of a real homogeneous space provided with an invariant flat connection.

Koszul's use of the affine representation of Lie groups and Lie algebras drew our attention, especially the similarities of his approach with that used by Jean-Marie Souriau in geometric mechanics in the framework of homogeneous symplectic manifolds. We then initiated explorations to make the bridge between Koszul and Souriau's works. We finally discovered that, in 1986, Koszul published this book "*Introduction to symplectic geometry*" following a Koszul lectured in English course in China. We also observed that this book analyzes in detail and develops Souriau's works on homogeneous symplectic manifolds in Geometric Mechanics by the means of ***the affine representation of Lie algebras and Lie groups***. Chuan Yu Ma writes in a review of this book in Chinese that "*This work coincided with developments in the field of analytical mechanics. Many new ideas have also been derived using a wide variety of notions of modern algebra, differential geometry, Lie groups, functional analysis, differentiable manifolds, and representation theory. [Koszul's book] emphasizes the differential-geometric and topological properties of symplectic manifolds. It gives a modern treatment of the subject that is useful for beginners as well as for experts*."

We then started an epistolary correspondence with Professor Koszul on Souriau's works and on the genesis of this book. In May 2015, questioning Koszul on Souriau's work on Geometric Mechanics and on Lie Group Thermodynamics, Koszul answered me "[*A l'époque où Souriau développait sa théorie, l'establishment avait tendance à ne pas y voir des avancées importantes. Je l'ai entendu exposer ses idées sur la thermodynamique mais je n'ai pas du tout réalisé à l'époque que la géométrie hessienne était en jeu.*] *At the time when Souriau was* *developing his theory, the establishment tended not to see significant progress. I heard him explaining his ideas on thermodynamics but I did not realize at the time that Hessian geometry was at stake*." In September 2016, I asked him about the origins of Lie Group and Lie Algebra Affine representations. Koszul informed me about the seminal work of Elie Cartan, who gave him lectures at ENS Ulm on homogeneous bounded domains, the germinal root of his Ph.D.: "*[Il y a là bien des choses que je voudrais comprendre (trop peut-être !), ne serait-ce que la relation entre ce que j'ai fait et les travaux de Souriau. Détecter l'origine d'une notion ou la première apparition d'un résultat est souvent difficile. Je ne suis certainement pas le premier à avoir utilisé des représentations affines de groupes ou d'algèbres de Lie. On peut effectivement imaginer que cela se trouve chez Elie Cartan, mais je ne puis rien dire de précis. A propos d'Elie Cartan: je n'ai pas été son élève. C'est Henri Cartan qui a été mon maître pendant mes années de thèse. En 1941 ou 42 j'ai entendu une brève série de conférences données par Elie à l'Ecole Normale et ce sont des travaux d'Elie qui ont été le point de départ de mon travail de thèse.] There are many things that I would like to understand (too much perhaps!), if only the relationship between what I did and the work of Souriau. Detecting the origin of a notion or the first appearance of a result is often difficult. I am certainly not the first to have used affine representations of groups or Lie algebras. We can imagine that it is Elie Cartan, but I cannot say anything specific. About Elie Cartan: I was not his student. It was Henri Cartan who was my master during my thesis years. In 1941 or 42, I heard a brief series of lectures given by Elie at the Ecole Normale and it was Elie's work that was the starting point of my thesis work.*"

After discovering the existence of Koszul's book, written in Chinese, based on a course "*Introduction to Symplectic Geometry*", given at Nakai, in which he made reference to Souriau's book and developed his main tools, we started to discuss its content. In January 2017, Koszul wrote me with his usual humility "*[Ce petit fascicule d'introduction à la géométrie symplectique a été rédigé par un assistant de Nankai qui avait suivi mon cours. Il n'y a pas eu de version initiale en français.] This small introductory booklet on symplectic geometry was written by a Nankai assistant who had taken my course. There was no initial version in French*." I asked him if he had a personal archive of this course. He answered "*[Je n'ai pas conservé de notes préparatoires à ce cours. Dites-moi à quelle adresse je puis vous envoyer un exemplaire du texte chinois.] I have not kept any preparatory notes for this course. Tell me where I can send you a copy of the Chinese text.*" Professor Koszul then sent me his last copy of this book in Chinese (Fig. 8).

I was not able to read the Chinese text, but I have observed in Chap. 4 "Symplectic G-spaces", and in Chap. 5 "Poisson Manifolds", that their equations include new original developments of Souriau's work on moment maps and affine representation of Lie Groups and Lie Algebra. More particularly, Koszul considered in great detail "non-equivariance" case of co-adjoint action on moment map, where I recovered Souriau's theorem. Koszul shows that when $(M; \omega)$ is a connected Hamiltonian G-space and μ a moment of the action of G, there exists an affine action of G on $\mathfrak{g}^*$ (dual Lie algebra), whose linear part is the coadjoint action, for which the moment μ is equivariant. Koszul developed Souriau's idea that this

Fig. 8 Koszul's original "little green" book "Introduction to Symplectic Geometry" in Chinese from his lecture at Nankai

affine action is obtained by modifying the coadjoint action by means of a closed cochain (called a cocycle by Souriau), and that $(M; \omega)$ is a G-Poisson space, making reference to Souriau's book for more details.

About collaboration between Koszul and Souriau and another potential lecture on Symplectic Geometry in Toulouse, Koszul informed me in February 2017 that: "*[J'ai plus d'une fois rencontré Souriau lors de colloques, mais nous n'avons jamais collaboré. Pour ce qui est de cette allusion à un "cours" donné à Toulouse, il y erreur. J'y ai peut être fait un exposé en 81, mais rien d'autre.] I have met Souriau more than once at conferences, but we have never collaborated. As for this allusion to a "course" given in Toulouse, there is an error. I could have made a presentation in 81, but nothing else*." Koszul admitted that he had no direct collaboration with Souriau: "*[Je ne crois pas avoir jamais parlé de ses travaux avec Souriau. Du reste j'avoue ne pas en avoir bien mesuré l'importance à l'époque] I do not think I ever talked to Souriau about his work. For the rest, I admit that I did not have a good idea of its importance at the time.*"

Considering the importance of this book for different communities, I tried to find an editor for its translation into English. By chance, I met Catriona Byrne from SPRINGER, when I gave a talk at IHES, invited by Pierre Cartier, on applications of Koszul and Souriau's work to Radar (concluded by beautiful pieces of music written by Julien Koszul, Jean-Louis' grandfather, performed by Bertrand Maury). With the perseverance of Michel Boyom, we convinced Professor Koszul to translate this book, proposing to contextualize it with regard to contemporary research trends in Geometric Mechanics, Lie Group Thermodynamics, and the Geometric Science of Information. Professors Marle and Boyom agreed to check the translation and help me to write the forewords.

Koszul's Book: A Joint Source of Geometric Heat Theory and Information Geometry

In the Foreword of this book, Koszul writes "*The development of analytical mechanics provided the basic concepts of symplectic structures. The term symplectic structure is due largely to analytical mechanics. But in this book, the applications of symplectic structure theory to mechanics is not discussed in any detail*." Koszul considers purely algebraic and geometric developments of Geometric/Analytic Mechanics developed during the 60s, in particular, Jean-Marie Souriau's works detailed in Chaps. 4 and 5. ***The originality of this book lies in the fact that Koszul develops new points of view, and demonstrations not initially considered by Souriau and later developed by the Geometric Mechanics community***.

Jean-Marie Souriau was the Creator of a new discipline called "*Mécanique Géométrique (Geometric Mechanics)*". Souriau observed that the collection of motions of a dynamical system is a manifold with an antisymmetric flat tensor that is a symplectic form where the structure contains all the pertinent information on the state of the system (positions, velocities, forces, etc.). Souriau said: "*[Ce que Lagrange a vu, que n'a pas vu Laplace, c'était la structure symplectique] What Lagrange saw, that Laplace didn't see, was the symplectic structure.*" Using the symmetries of a symplectic manifold, Souriau introduced a mapping which he called the "*moment map*", which takes its values in a space attached to the group of symmetries (in the dual space of its Lie algebra). The moment map allows one to build conserved quantities for the group action, generalizing the classical notions of linear and angular momentum. Souriau associated to this moment maps the notion of symplectic cohomology, linked to the fact that such a moment is defined up to an additive constant that brings into play an algebraic mechanism (called cohomology). Souriau proved that the moment map is a constant of the motion and provided a geometric generalization of Emmy Noether's invariant theorem (invariants of E. Noether's theorem are the components of the moment map; but where Noether's approach is purely algebraic, Souriau's approach gives geometric roots and meanings to these invariants). Souriau has defined in a geometrical way the Noetherian symmetries using the Lagrange–Souriau 2-form with the moment map. Influenced by François Gallissot (Souriau and Galissot both attended ICM'54 in Moscow. Did they discuss this point?), Souriau introduced in Mechanics the Lagrange 2-form, recovering Lagrange's seminal ideas. Motivated by the need to give a coordinate-independent formulation of the variational principles, inspired by Henri Poincaré and Elie Cartan who introduced a differential 1-form instead of the Lagrangian, Souriau introduced the Lagrange 2-form as the exterior differential of the Poincaré–Cartan 1-form, and obtained the phase space as a symplectic manifold. Souriau proposed to consider this Lagrange 2-form as the fundamental structure for Lagrangian system and not the classical Lagrangian function or the Poincaré–Cartan 1-form. This 2-form is called the Lagrange-Souriau 2-form and is the exterior derivative of the Lepage form (the Poincaré–Cartan form is a first-order Lepage form). This structure is developed in Koszul's book, where the authors show that when $(M; \omega)$ is an exact symplectic manifold (when there exists a 1-form α on M such that $\omega = -d\alpha$), and that a symplectic action leaves not only ω, but α invariant, this action is strongly Hamiltonian ($(M; \omega)$ is a g-Poisson space). Koszul shows that a symplectic action of a Lie algebra $\mathfrak{g}^*$ on an exact symplectic manifold $(M; \omega = -d\alpha)$ that leaves invariant not only ω, but also α, is strongly Hamiltonian.

In this book, in Chap. 4, Koszul defines symplectic G-space as a symplectic manifold $(M; \omega)$ on which a Lie group G acts by a symplectic action (an action which leaves unchanged the symplectic form ω). Koszul then introduces and develops properties of the moment map μ (Souriau's invention) of a Hamiltonian action of the Lie algebra $\mathfrak{g}^*$. He also defines the Souriau 2-cocycle, considering that the difference of two moments of the same Hamiltonian action is a locally constant function on M, showing that when μ is a moment map, for every pair (a, b) of elements of $\mathfrak{g}^*$, the function $c_\mu(a,b) = \{\langle \mu, a\rangle, \langle \mu, b\rangle\} - \langle \mu, \{a, b\}\rangle$ is locally constant on M, defining an antisymmetric bilinear map of $\mathfrak{g} \times \mathfrak{g}$ in $H^0(M; R)$ which satisfies Jacobi's identity. ***This is the 2-cocycle introduced by Souriau in Geometric Mechanics, which plays a fundamental role in Souriau's Lie Groups Thermodynamics where it is used to define an extension of the Fisher Metric from Information Geometry (what we will call the Souriau–Fisher metric in the following)***.

To highlight the importance of Koszul's book, we will illustrate its links of the detailed tools, including demonstrations or original Koszul extensions, with Souriau's Lie Group Thermodynamics, whose applications range from statistical physics to machine learning in Artificial Intelligence. Koszul originally developed Souriau's model, in the case of non-equivariance, of the action of the group G on the moment map. As explained in [51] by Thomas Delzant at the 2010 CIRM conference "Action Hamiltoniennes: invariants et classification", organized with Michel Brion: "*The definition of the moment map is due to Jean-Marie Souriau…. In the book of Souriau we find a proof of the proposition: the map J is equivariant for an affine action of G on* $\mathfrak{g}^*$ *whose linear part is* Ad^**…. In Souriau's book, we can also find a study of the non-equivariant case and its applications to classical and quantum mechanics. In the case of the Galileo group operating in the phase* *space of space-time, obstruction to equivariance (a class of cohomology) is interpreted as the inert mass of the object under study. We can uniquely define the moment map up to an additive constant of integration that can always be chosen to make the moment map equivariant (a moment map is G-equivariant when G acts on* $\mathfrak{g}^*$ *via the coadjoint action) if the group is compact or semi-simple. In 1969, Souriau considered the non-equivariant case where the coadjoint action must be modified to make the map equivariant under a 1-cocycle on the group with values in the dual Lie algebra* $\mathfrak{g}^*$.”

The concept and seminal idea of the moment map appeared in the second volume of Sophus Lie’s book, published in 1890, developed for homogeneous canonical transformations. Professor Marsden summarized the development of this concept by Jean-Marie Souriau and Bertram Kostant based on their two testimonials: “*In Kostant’s 1965 Phillips lectures at Haverford, and in the 1965 U.S.–Japan Seminar, Kostant introduced the momentum map to generalize a theorem of Wang and thereby classified all homogeneous symplectic manifolds; this is called today ‘Kostant’s coadjoint orbit covering theorem’… . Souriau introduced the momentum map in his 1965 Marseille lecture notes and put it in print in 1966. The momentum map finally got its formal definition and its name, based on its physical interpretation, by Souriau in 1967. Souriau also studied its properties of equivariance, and formulated the coadjoint orbit theorem. The momentum map appeared as a key tool in Kostant’s quantization lectures in 1970* [52]*, and Souriau discussed it at length in 1970 in his book [31]. Kostant and Souriau realized its importance for linear representations, a fact apparently not foreseen by Lie*.” Souriau’s book is referred to be published by Dunod in 1970, but Souriau manuscript and book was available as soon as 1969. Incidentally, Jean-Louis Koszul knew Souriau and Kostant’s work very well, and as early as 1958, Koszul made a survey of Kostant’s first work at the Bourbaki seminar [53].

In 1969, Souriau also introduced the concept of a coadjoint action of a group on its moment space in the framework of Thermodynamics, based on the orbit method, which allows one to define physical observables like energy, heat, and momentum or moment as pure geometrical objects. In the first step to establishing a new foundations of thermodynamics, Souriau defined a Gibbs canonical ensemble on a symplectic manifold M for a Lie group action on M. In classical statistical mechanics, a state is given by the solution of Liouville’s equation on the phase space, the partition function. As symplectic manifolds have a completely continuous measure, invariant under diffeomorphisms (the Liouville measure λ), Souriau proved that when statistical states are Gibbs states (as generalized by Souriau), they are the product of the Liouville measure by the scalar function given by the generalized partition function $e^{\Phi(\beta)-\langle\beta,U(\xi)\rangle}$defined by the energy U (defined in the dual of the Lie algebra of this dynamical group) and the geometric temperature β, where Φ is a normalizing constant such that the mass of probability is equal to 1, $\Phi(\beta) = -\log\int\limits_M e^{-\langle\beta,U(\xi)\rangle}d\lambda$. Souriau then generalizes the Gibbs equilibrium state to all symplectic manifolds that have a dynamical group. Souriau has observed that theory to dynamical groups in Physics (Galileo or Poincaré groups), the symmetry will be broken. For each temperature β, element of the Lie algebra $\mathfrak{g}$, Souriau introduced a tensor $\tilde{\Theta}_\beta$, equal to the sum of the cocycle $\tilde{\Theta}$ and the heat coboundary (with [.,.] Lie bracket):

$$\tilde{\Theta}_\beta(Z_1, Z_2) = \tilde{\Theta}(Z_1, Z_2) + \langle Q, ad_{Z_1}(Z_2) \rangle \tag{2}$$

This tensor $\tilde{\Theta}_\beta$ has the following properties: $\tilde{\Theta}(X, Y) = \langle \Theta(X), Y \rangle$, where the map Θ is the symplectic one-cocycle of the Lie algebra $\mathfrak{g}$ with values in $\mathfrak{g}^*$, with $\Theta(X) = T_e\theta(X(e))$, where θ is the one-cocycle of the Lie group G. $\tilde{\Theta}(X, Y)$ is constant on M and the map $\tilde{\Theta}(X, Y) : \mathfrak{g} \times \mathfrak{g} \to \Re$ is a skew-symmetric bilinear form, and is called the ***symplectic two-cocycle of the Lie algebra*** $\mathfrak{g}$ associated to the *moment map* J, with the following properties:

$$\tilde{\Theta}(X, Y) = J_{[X,Y]} - \{J_X, J_Y\} \text{ with } J \text{ the moment map} \tag{3}$$

$$\tilde{\Theta}([X, Y], Z) + \tilde{\Theta}([Y, Z], X) + \tilde{\Theta}([Z, X], Y) = 0 \tag{4}$$

where J_X linear map from $\mathfrak{g}$ to a differential function on $M : \mathfrak{g} \to C^\infty(M, R), X \to J_X$ and the associated differentiable map J, called the moment(um) map, is defined by

$$J : M \to \mathfrak{g}^*,\ x \mapsto J(x) \text{ such that } J_X(x) = \langle J(x), X \rangle, X \in \mathfrak{g} \tag{5}$$

The geometric temperature, an element of the algebra $\mathfrak{g}$, is in the kernel of the tensor $\tilde{\Theta}_\beta$: $\beta \in Ker\,\tilde{\Theta}_\beta$ such that

$$\tilde{\Theta}_\beta(\beta, \beta) = 0,\ \forall \beta \in \mathfrak{g} \tag{6}$$

The following symmetric tensor $\mathfrak{g}_\beta([\beta, Z_1], [\beta, Z_2]) = \tilde{\Theta}_\beta(Z_1, [\beta, Z_2])$, defined on all values of $ad_\beta(.) = [\beta, .]$, is positive definite and defines an extension of the classical Fisher metric in Information Geometry (as the Hessian of the logarithm of the partition function):

$$g_\beta([\beta, Z_1], Z_2) = \tilde{\Theta}_\beta(Z_1, Z_2),\ \forall Z_1 \in g,\ \forall Z_2 \in \mathrm{Im}\big(ad_\beta(.)\big) \tag{7}$$

where

$$g_\beta(Z_1, Z_2) \geq 0,\ \forall Z_1, Z_2 \in \mathrm{Im}\big(ad_\beta(.)\big) \tag{8}$$

***These equations are universal*, because they are not dependent on the symplectic manifold but only on the dynamical group G, the symplectic two-cocycle Θ, the temperature β and the heat Q. Souriau called this "*Lie group thermodynamics*".**

This antisymmetric bilinear map (7) is precisely the mathematical object introduced in Chap.4 of Koszul's book by:

$c_\mu(a,b) = \{\langle \mu, a\rangle, \langle \mu, b\rangle\} - \langle \mu, \{a,b\}\rangle$.

For any moment map μ, Koszul defines the skew-symmetric bilinear form $\mathrm{c}_\mu(\mathrm{a,b})$ on a Lie algebra by:

$$\mathrm{c}_\mu(\mathrm{a,b}) = \left\langle \mathrm{d}\theta_\mu(a), b \right\rangle,\ a,b \in \mathfrak{g} \tag{9}$$

Koszul observes that if we use:

$$\begin{aligned}\theta_\mu(st) &= \mu(stx) - Ad^*_{st}\mu(x) = \theta_\mu(s) + Ad^*_s\mu(tx) - Ad^*_s Ad^*_t\mu(x)\\ &= \theta_\mu(s) + Ad^*_s\theta_\mu(t)\end{aligned} \tag{10}$$

by developing $d\mu(ax) = {}^t ad_a\mu(x) + d\theta_\mu(a),\ x \in M, a \in \mathfrak{g}$, he obtains:

$$\begin{aligned}\langle d\mu(ax), b\rangle &= \langle \mu(x), [a,b]\rangle + \left\langle \mathrm{d}\theta_\mu(a), b\right\rangle\\ &= \{\langle \mu, \mathrm{a}\rangle, \langle \mu, b\rangle\}(x),\ x \in M, a, b \in \mathfrak{g}\end{aligned} \tag{11}$$

We then have:

$$\mathrm{c}_\mu(\mathrm{a,b}) = \{\langle \mu, \mathrm{a}\rangle, \langle \mu, b\rangle\} - \langle \mu, [a,b]\rangle = \left\langle \mathrm{d}\theta_\mu(a), b\right\rangle,\ a,b \in \mathfrak{g} \tag{12}$$

and the property:

$$\mathrm{c}_\mu([\mathrm{a,b}], \mathrm{c}) + c_\mu([b,c], a) + c_\mu([c,a], b) = 0,\ a,b,c \in \mathfrak{g} \tag{13}$$

Koszul concludes by observing that if the moment map is transformed as $\mu' = \mu + \phi$ then we have:

$$c_{\mu'}(a,b) = c_\mu(a,b) - \langle \phi, [a,b]\rangle \tag{14}$$

Finally, using $\mathrm{c}_\mu(\mathrm{a,b}) = \{\langle \mu, \mathrm{a}\rangle, \langle \mu, b\rangle\} - \langle \mu, [a,b]\rangle = \left\langle \mathrm{d}\theta_\mu(a), b\right\rangle,\ a,b \in \mathfrak{g}$, Koszul highlights the property that:

$$\{\mu^*(a), \mu^*(b)\} = \{\langle \mu, a\rangle, \langle \mu, b\rangle\} = \mu^*\big([a,b] + c_\mu(a,b)\big) = \mu^*\{a,b\}_{c_\mu} \tag{15}$$

In Chap. 4, Koszul introduces the equivariance of the moment map μ. Based on the definitions of the adjoint and coadjoint representations of a Lie group or a Lie algebra, Koszul proves that when $(M;\ \omega)$ is a connected Hamiltonian G-space and $\mu : M \to \mathfrak{g}^*$ a moment of the action of G, there exists an affine action of G on $\mathfrak{g}^*$, whose linear part is the coadjoint action, for which the moment μ is equivariant. This affine action is obtained by modifying the coadjoint action by means of a cocycle. This notion is also developed in Chap. 5 for Poisson manifolds.

Defining the classical operation
$Ad_s a = sas^{-1},\ s \in G, a \in \mathfrak{g},\ ad_a b = [a,b], a \in \mathfrak{g}, b \in \mathfrak{g}$ and $Ad_s^* = {}^t Ad_{s^{-1}},\ s \in G$ with classical properties:

$$Ad_{\exp a} = \exp(-ad_a),\ a \in g \text{ or } Ad^*_{\exp a} = \exp{}^t(ad_a), a \in \mathfrak{g} \tag{16}$$

Koszul considers:

$$x \mapsto sx,\ x \in M, \mu : M \to \mathfrak{g}^* \tag{17}$$

from which he obtains:

$$\langle d\mu(v), a\rangle = \omega(ax, v) \tag{18}$$

Koszul then studies $\mu \circ s_M - Ad_s^* \circ \mu : M \to \mathfrak{g}^*$ and develops:

$$d\left\langle Ad_s^* \circ \mu, a\right\rangle = \left\langle Ad_s^* d\mu, a\right\rangle = \langle d\mu, Ad_{s^{-1}} a\rangle \tag{19}$$

$$\begin{aligned}\langle d\mu(v), Ad_{s^{-1}} a\rangle &= \omega\left(s^{-1}asx, v\right) = \omega(asx, sv) \\ &= \langle d\mu(sv), a\rangle = (d\langle \mu \circ s_M, a\rangle)(v)\end{aligned} \tag{20}$$

So $d\left\langle Ad_s^* \circ \mu, a\right\rangle = d\langle \mu \circ s_M, a\rangle$ then proving that

$$d\left\langle \mu \circ s_M - Ad_s^* \circ \mu, a\right\rangle = 0 \tag{21}$$

Koszul considers the cocycle given by $\theta_\mu(s) = \mu(sx) - Ad_s^* \mu(x),\ s \in G$ and observes that:

$$\theta_\mu(st) = \theta_\mu(s) - Ad_s^* \theta_\mu(\mathrm{t})\ ,\ s, t \in G \tag{22}$$

From this action of the group on the dual Lie algebra:

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, (s, \xi) \mapsto s\xi = Ad_s^* \xi + \theta_\mu(s) \tag{23}$$

Koszul introduces the following properties:

$$\mu(sx) = s\mu(x) = Ad_s^* \mu(x) + \theta_\mu(s),\ \forall s \in G, x \in M \tag{24}$$

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, (e, \xi) \mapsto e\xi = Ad_e^* \xi + \theta_\mu(e) = \xi + \mu(x) - \mu(x) = \xi \tag{25}$$

$$\begin{aligned}(s_1 s_2)\xi &= Ad^*_{s_1 s_2} \xi + \theta_\mu(s_1 s_2) = Ad^*_{s_1} Ad^*_{s_2} \xi + \theta_\mu(s_1) + Ad^*_{s_1} \theta_\mu(s_2) \\ (s_1 s_2)\xi &= Ad^*_{s_1}\left(Ad^*_{s_2} \xi + \theta_\mu(s_2)\right) + \theta_\mu(s_1) = s_1(s_2 \xi),\ \forall s_1, s_2 \in G, \xi \in \mathfrak{g}^*\end{aligned} \tag{26}$$

Koszul's study of the moment map μ equivariance, and the existence of an affine action of G on $\mathfrak{g}^*$, whose linear part is the coadjoint action, for which the moment μ is equivariant, is at the cornerstone of Souriau's Theory of

Geometric Mechanics and Lie Group Thermodynamics. We illustrate its importance by giving Souriau's theorem of Lie Group Thermodynamics:

Theorem (Souriau's Theorem of Lie Group Thermodynamics). *Let Ω be the largest open proper subset of $\mathfrak{g}$, the Lie algebra of G, such that $\int_M e^{-\langle \beta, U(\xi)\rangle} d\lambda$ and $\int_M \xi . e^{-\langle \beta, U(\xi)\rangle} d\lambda$ are convergent integrals, this set Ω is convex and is invariant under every transformation $Ad_g(.)$. Then, the fundamental equations of Lie group thermodynamics are given by the action of the group:*

- *Action of the Lie group on the Lie algebra:* $\beta \to Ad_g(\beta)$ (27)
- *Characteristic function after Lie group action:* $\Phi \to \Phi - \langle \theta(g^{-1}), \beta \rangle$ (28)
- *Invariance of entropy with respect to the action of the Lie group:* $s \to s$ (29)
- *Action of the Lie group on geometric heat:* $Q \to a(g, Q) = Ad_g^*(Q) + \theta(g)$ (30)

Souriau's equations of Lie group thermodynamics, related to the moment map μ equivariance, and the existence of an affine action of G on $\mathfrak{g}^*$, whose linear part is the coadjoint action, for which the moment μ is equivariant, are summarized in the following figures (Figs. 9 and 10).

We finally observe that ***the Koszul antisymmetric bilinear map*** $c_\mu(a, b) = \{\langle \mu, a\rangle, \langle \mu, b\rangle\} - \langle \mu, \{a, b\}\rangle$ ***is equal to Souriau's Riemannian metric***, introduced by means of a symplectic cocycle. We have observed that this metric is a generalization of the Fisher metric from Information Geometry that we call the Souriau–Fisher metric, defined as a hessian of the partition function logarithm $g_\beta = -\frac{\partial^2 \Phi}{\partial \beta^2} = \frac{\partial^2 \log \psi_\Omega}{\partial \beta^2}$ as in classical information geometry. We can establish the equality of two terms, between Souriau's definition based on the Lie group cocycle Θ and parameterized by "geometric heat" Q (element of dual Lie algebra) and "geometric temperature" β (element of the Lie algebra) and the hessian of the characteristic function $\Phi(\beta) = -\log \psi_\Omega(\beta)$ with respect to the variable β:

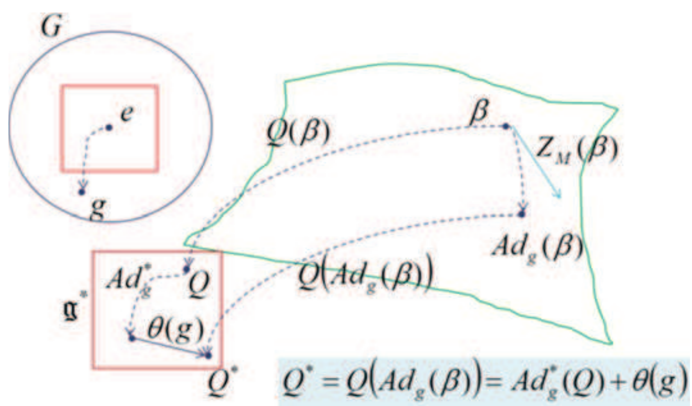


Fig. 9 Broken symmetry on the geometric heat Q due to the adjoint action of the group on the temperature β as an element of the Lie algebra

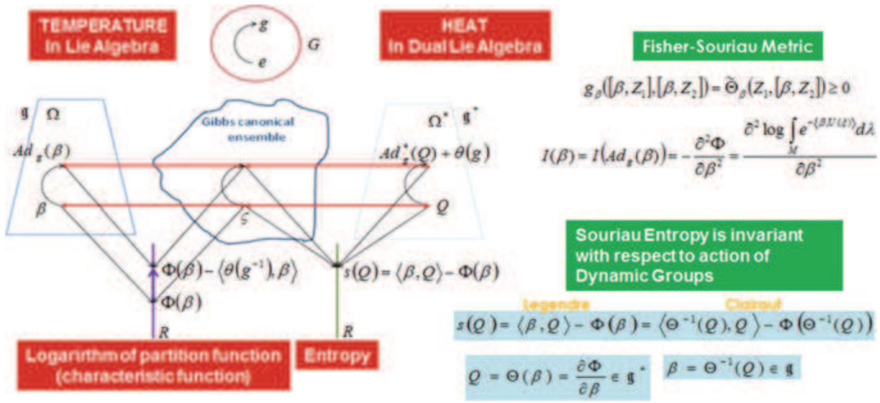


Fig. 10 Sourau's global scheme of Lie group thermodynamics

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \langle\Theta(Z_1),[\beta,Z_2]\rangle + \langle Q,[Z_1,[\beta,Z_2]]\rangle = \frac{\partial^2 \log\psi_\Omega}{\partial\beta^2} \tag{31}$$

If we differentiate this relation of Souriau's theorem $Q\big(Ad_g(\beta)\big) = Ad_g^*(Q) + \theta(g)$, we obtain:

$$\frac{\partial Q}{\partial\beta}(-[Z_1,\beta],.) = \tilde{\Theta}(Z_1,[\beta,.]) + \langle Q, Ad_{.Z_1}([\beta,.])\rangle = \tilde{\Theta}_\beta(Z_1,[\beta,.]) \tag{32}$$

$$-\frac{\partial Q}{\partial\beta}([Z_1,\beta],Z_2) = \tilde{\Theta}(Z_1,[\beta,Z_2]) + \langle Q, Ad_{.Z_1}([\beta,Z_2])\rangle = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2]) \tag{33}$$

$$\Rightarrow -\frac{\partial Q}{\partial\beta} = g_\beta([\beta,Z_1],[\beta,Z_2]) \tag{34}$$

The Fisher metric $I(\beta) = -\frac{\partial^2\Phi(\beta)}{\partial\beta^2} = -\frac{\partial Q}{\partial\beta}$ has been considered by Souriau as a *generalization of* "*heat capacity*". Souriau called it the "*geometric capacity*".

Affine Representation of Lie Groups and Lie Algebras: Koszul and Souriau's Pillars Based on Elie Cartan's Seminal Work

The affine representation of Lie groups/algebras used by Koszul in this book has been intensively studied by Souriau, who called the mechanics deduced from this model "affine mechanics". We will explain notions such as moment map, equivariance of moment map, and cocycles through this notion of affine representation of Lie Groups and Lie Algebras. Previously, we presented Souriau's work on the affine representation of a Lie group used to elaborate Lie Group Thermodynamics. Here, we will present Koszul's original approach to the affine representation of Lie groups and Lie algebras in a finite-dimensional vector space, seen as special examples of actions.

Since the work of Henri Poincaré and Elie Cartan, the theory of differential forms has become an essential instrument of modern differential geometry, used by Souriau to identify the space of motions as a symplectic manifold. However, as observed by Paulette Libermann, except for Henri Poincaré, who wrote shortly before his death a report on the work of Elie Cartan during his application to the Sorbonne University, the French mathematicians did not see the importance of Cartan's breakthroughs. Souriau followed lectures of Elie Cartan in 1945. Elie Cartan's second student was Jean-Louis Koszul. Koszul studied symmetric homogeneous spaces and defined relations between invariant flat affine connections to affine representations of Lie algebras, and characterized invariant Hessian metrics by affine representations of Lie algebras. Koszul provided a correspondence between symmetric homogeneous spaces with invariant Hessian structures by using affine representations of Lie algebras and proved that a simply connected symmetric homogeneous space with invariant Hessian structure is a direct product of a Euclidean space and a homogeneous self-dual regular convex cone. Let G be a connected Lie group and let G/K be a homogeneous space on which G acts effectively. Koszul gave a bijective correspondence between the set of G-invariant flat connections on G/K, and the set of a certain class of affine representations of the Lie algebra of G. Koszul's main theorem is: let G/K be a homogeneous space of a connected Lie group G and let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, assuming that G/K is endowed with a G-invariant flat connection, then $\mathfrak{g}$ admits an affine representation (f,q) on the vector space E. Conversely, suppose that G is simply connected and that g is endowed with an affine representation, then G/K admits a G-invariant flat connection.

In the foregoing, the basic tool studied by Koszul is the affine representation of Lie algebras and Lie groups. To study these structures, Koszul introduced the following developments. Let Ω be a convex domain on R^n without any straight lines, and an associated convex cone $V(\Omega) = \{(\lambda x, x) \in R^n \times R/x \in \Omega, \lambda \in R^+\}$, then there exists an affine embedding:

$$\ell : x \in \Omega \mapsto \begin{bmatrix} x \\ 1 \end{bmatrix} \in V(\Omega) \tag{35}$$

If we consider the group homomorphism η of $A(n,R)$ to $GL(n+1,R)$ given by:

$$s \in A(n,R) \mapsto \begin{bmatrix} \mathbf{f}(s) & \mathbf{q}(s) \\ 0 & 1 \end{bmatrix} \in GL(n+1,R) \tag{36}$$

and the affine representation of the Lie algebra:

$$\begin{bmatrix} f & q \\ 0 & 0 \end{bmatrix} \tag{37}$$

where $A(n,R)$ is the group of all affine representations of R^n, we have $\eta(G(\Omega)) \subset G(V(\Omega))$ and the pair (η, ℓ) comprising the homomorphism $\eta : G(\Omega) \to G(V(\Omega))$ and the map $\ell : \Omega \to V(\Omega)$ is equivariant.

Observing Koszul's affine representations of Lie algebras and Lie groups, we have to consider a convex Lie group G and a real or complex vector space E of finite dimension, Koszul introduced an affine representation of G in E such that:

$$\begin{gathered} E \to E \\ a \mapsto sa \quad \forall s \in G \end{gathered} \tag{38}$$

is an affine representation. We set $A(E)$ the set of all affine transformation of a real vector space E, a Lie group called affine representation group of E. The set $GL(E)$ of all regular linear representation of E, a subgroup of $A(E)$. We define a linear representation of G in $GL(E)$:

$$\begin{gathered} \mathbf{f}\colon G \to GL(E) \\ s \mapsto \mathbf{f}(s)a = sa - so\,\forall a \in E \end{gathered} \tag{39}$$

and a map from G to E:

$$\begin{gathered} \mathbf{q}\colon G \to E \\ s \mapsto \mathbf{q}(s) = so\,\forall s \in G \end{gathered} \tag{40}$$

then, we have $\forall s,\ t \in G$:

$$\mathbf{f}(s)\mathbf{q}(t) + \mathbf{q}(s) = \mathbf{q}(st) \tag{41}$$

Since $\mathbf{f}(s)\mathbf{q}(t) + \mathbf{q}(s) = s\mathbf{q}(t) - so + so = s\mathbf{q}(t) = sto = \mathbf{q}(st)$.

Conversely, if a map $\mathbf{q}$ from G to E and a linear representation $\mathbf{f}$ from G to $GL(E)$ satisfy the previous equation, then we can define an affine representation from G in E, written $(\mathbf{f}, \mathbf{q})$:

$$Aff(s) : a \mapsto sa = \mathbf{f}(s)a + \mathbf{q}(s),\ \forall s \in G, \forall a \in E \tag{42}$$

The condition $\mathbf{f}(s)\mathbf{q}(t) + \mathbf{q}(s) = \mathbf{q}(st)$ is equivalent to the statement that the following mapping is a homomorphism:

$$Aff : s \in G \mapsto Aff(s) \in A(E) \tag{43}$$

We denote by f the affine representation of the Lie algebra $\mathfrak{g}$ of G, defined by $\mathbf{f}$, and q the restriction to $\mathfrak{g}$ of the differential of $\mathbf{q}$ (f and q are the differentials of $\mathbf{f}$ and $\mathbf{q}$, respectively). Koszul proved the following equation:

$$\begin{aligned} f(X)q(Y) - f(Y)q(X) = q([X, Y]) \quad \forall X,\ Y \in \mathfrak{g} \\ \text{and} \quad f : \mathfrak{g} \to gl(E) \quad \text{and} \quad q : \mathfrak{g} \mapsto E \end{aligned} \tag{44}$$

and $gl(E)$ is the set of all linear endomorphisms of E, the Lie algebra of $GL(E)$. We use the assumption that:

$$q(Ad_s Y) = \left.\frac{d\mathbf{q}(s.e^{tY}.s^{-1})}{dt}\right|_{t=0} = \mathbf{f}(s)f(Y)\mathbf{q}(s^{-1}) + \mathbf{f}(s)q(Y) \tag{45}$$

We then obtain:

$$q([X, Y]) = \left.\frac{d\mathbf{q}(Ad_{e^{tX}} Y)}{dt}\right|_{t=0} = f(X)q(Y)\mathbf{q}(e) + \mathbf{f}(e)f(Y)(-q(X)) + f(X)q(Y) \tag{46}$$

where e is neutral element of G. Since $\mathbf{f}(e)$ is the identity map and $\mathbf{q}(e) = 0$, we have the equality:

$$f(X)q(Y) - f(Y)q(X) = q([X, Y]) \tag{47}$$

A pair (f, q) comprising a linear representation of f of a Lie algebra $\mathfrak{g}$ on E and a linear map q from $\mathfrak{g}$ in E is an affine representation of $\mathfrak{g}$ in E if it satisfies:

$$f(X)q(Y) - f(Y)q(X) = q([X, Y]) \tag{48}$$

Conversely, if we assume that $\mathfrak{g}$ has an affine representation (f, q) on E, by using the coordinate systems $\{x^1, \ldots, x^n\}$ on E, we can express the affine map $v \mapsto f(X)v + q(Y)$ by a matrix representation of size $(n+1) \times (n-1)$:

$$aff(X) = \begin{bmatrix} f(X) & q(X) \\ 0 & 0 \end{bmatrix} \tag{49}$$

where $f(X)$ is a matrix of size $n \times n$ and $q(X)$ a vector of size n.

$X \mapsto aff(X)$ is an injective homomorphism of the Lie algebra $\mathfrak{g}$ to the Lie algebra of $(n+1) \times (n+1)$ matrices, $gl(n+1, R)$:

$$\begin{aligned} \mathfrak{g} &\to gl(n+1, R) \\ X &\mapsto aff(X) \end{aligned} \tag{50}$$

Writing $\mathfrak{g}_{aff} = aff(\mathfrak{g})$, we denote by G_{aff} the linear Lie subgroup of $GL(n+1, R)$ generated by $\mathfrak{g}_{aff}$. An element of $s \in G_{aff}$ may be expressed by:

$$Aff(s) = \begin{bmatrix} \mathbf{f}(s) & \mathbf{q}(s) \\ 0 & 1 \end{bmatrix} \quad (51)$$

Let M_{aff} be the orbit of G_{aff} from the origin o, then $M_{aff} = \mathbf{q}(G_{aff}) = G_{aff}/K_{aff}$, where $K_{aff} = \{s \in G_{aff}/\mathbf{q}(s) = 0\} = Ker(\mathbf{q})$.

We can give as an example the following case. Let Ω be a convex domain in R^n without any straight lines. We define the cone $V(\Omega)$ in $R^{n+1} = R^n \times R$ by $V(\Omega) = \{(\lambda x, x) \in R^n \times R/x \in \Omega,\ \lambda \in R^+\}$. Then, there is an affine embedding:

$$\ell : x \in \Omega \mapsto \begin{bmatrix} x \\ 1 \end{bmatrix} \in V(\Omega) \quad (52)$$

Let η be the group homomorphism of $A(n,R)$ to $GL(n+1,R)$ given by:

$$s \in A(n,R) \mapsto \begin{bmatrix} \mathbf{f}(s) & \mathbf{q}(s) \\ 0 & 1 \end{bmatrix} \in GL(n+1,R) \quad (53)$$

where $A(n,R)$ is the group of all affine transformations in R^n. We have $\eta(G(\Omega)) \subset G(V(\Omega))$ and the pair (η, ℓ) comprising the homomorphism $\eta : G(\Omega) \to G(V(\Omega))$ and the map $\ell : \Omega \to V(\Omega)$ are equivariant:

$$\ell \circ s = \eta(s) \circ \ell \text{ and } d\ell \circ s = \eta(s) \circ d\ell \quad (54)$$

In Table 1, we compare the affine representations of Lie groups and Lie algebras according to each of Souriau and Koszul's approaches:

Table 1 Table comparing Souriau and Koszul's affine representations of Lie groups and Lie algebras

Souriau's model of affine representations of Lie groups and algebras (using the notation of Libermann–Marle)	Koszul's model of affine representations of Lie groups and algebras (using Koszul's notations)
$A(g)(x) = R(g)(x) + \theta(g)$ with $g \in G, x \in E$ $R : G \to GL(E)$ and $\theta : G \to E$	$Aff(s) : a \mapsto sa = \mathbf{f}(s)a + \mathbf{q}(s)\ \forall s \in G,\ \forall a \in E$ $\mathbf{f}$: $G \to GL(E)$ $s \mapsto \mathbf{f}(s)a = sa - so\ \forall a \in E$ $\mathbf{q}$: $G \to E$ $s \mapsto \mathbf{q}(s) = so\ \forall s \in G$
$\theta(gh) = R(g)(\theta(h)) + \theta(g)$ with $g, h \in G$ $\theta : G \to E$ is a one-cocycle of G with values in E,	$\mathbf{q}(st) = \mathbf{f}(s)\mathbf{q}(t) + \mathbf{q}(s)$
$a(X)(x) = r(X)(x) + \Theta(X)$ with $X \in \mathfrak{g}, x \in E$ The linear map $\Theta : \mathfrak{g} \to E$ is a one-cocycle of G with values in E: $\Theta(X) = T_e\theta(X(e)),\ X \in \mathfrak{g}$	$v \mapsto f(X)v + q(Y)$ f and q are the differentials of $\mathbf{f}$ and $\mathbf{q}$, respectively

(continued)

Table 1 (continued)

Souriau's model of affine representations of Lie groups and algebras (using the notation of Libermann–Marle)	Koszul's model of affine representations of Lie groups and algebras (using Koszul's notations)
$\Theta([X,Y]) = r(X)(\Theta(Y)) - r(Y)(\Theta(X))$	$q([X,Y]) = f(X)q(Y) - f(Y)q(X)\,\forall X,Y \in \mathfrak{g}$ with $f : \mathfrak{g} \to gl(E)$ and $q : \mathfrak{g} \mapsto E$
-	$aff(X) = \begin{bmatrix} f(X) & q(X) \\ 0 & 0 \end{bmatrix}$
-	$Aff(s) = \begin{bmatrix} \mathbf{f}(s) & \mathbf{q}(s) \\ 0 & 1 \end{bmatrix}$

Conclusion

Koszul's book, which uses a purely algebraic and geometric approach to reinforce Souriau's founding work in geometric mechanics, introducing original developments and proofs, is of major interest to various communities. In the first place, the community of physicists can find the mathematical foundations for analytical mechanics, but also for Lie Group Thermodynamics (the covariant theory of classical thermodynamics introduced by Souriau in parallel with geometric mechanics, but ignored by the great majority of the community). Second, this book is of great interest for the emerging field known as the "Geometric Science of Information", in which the generalization of the Fisher metric is at the heart of the extension of classical tools of Machine Learning and Artificial Intelligence to deal with more abstract objects living in homogeneous manifolds, groups, and structured matrices. Future of Information Geometry and Artificial Intelligence should be based on the pillars developed in Koszul's book. The "Geometric Science of Information" community (GSI, www.gsi2017.org) has lost a mathematician of great value, who enlightened his views by the depth of his thought.

> *… et sic matheseos demonstrationes cum aleae incertitudine jugendo, et quae contraria videntur conciliando, ab utraque nominationem suam accipiens, stupendum hunc titulum jure sibi arrogat:* ***Aleae Geometria***
>
> *… par l'union ainsi réalisée entre les démonstrations des mathématiques et l'incertitude du hasard, et par la conciliation entre les contraires apparents, elle peut tirer son nom de part et d'autre et s'arroger à bon droit ce titre étonnant:* ***Géométrie du Hasard***
>
> *… by the union thus achieved between the demonstrations of mathematics and the uncertainty of chance, and by the conciliation between apparent opposites, it can take its name from both sides and arrogate to right this amazing title:* ***Geometry of Chance***
> Blaise Pascal—ALEAE GEOMETRIA: De compositione aleae in ludis ipsi subjectis, in Celeberrimae matheseos Academiae Parisiensi, 1654

Limours, France Frédéric Barbaresco
June 2018

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Foreword 3

I was very pleased when Frédéric Barbaresco gave me the opportunity to write a short presentation for this book. Jean-Louis Koszul, recently deceased, was a highly distinguished mathematician whom I respected and admired. It is for me an honor to introduce a book stemming from the lectures he taught in 1983. In this Foreword, I am going to describe the content of this book and highlight its originality. Then I will say a few words about some developments in Symplectic and Poisson geometry, of which I am aware, not discussed in this book, often because they appeared after 1983.

The Content and Originality of this Book

The first chapter offers a very nice presentation of the main properties of symplectic vector spaces. The framework used at the beginning, slightly more general than that usually considered, is that of a finite-dimensional vector space V over an arbitrary field k, endowed with a skew-symmetric bilinear form ω whose kernel may not be $\{0\}$. Orthogonality with respect to ω is defined, as well as isotropic, coisotropic, Lagrangian, and symplectic vector subspaces, and their main properties are proven. Then it is assumed that $\ker\omega = \{0\}$, which means that (V, ω) is a symplectic vector space. *Symplectic bases* (often called, in other texts, *canonical bases* or *Darboux bases*) are defined. A long section is devoted to the canonical representation of the Lie algebra $sl(2, k)$ in the graded vector space of skew-symmetric multilinear forms on V. The last two sections of the chapter deal with the linear symplectic group and with adapted complex structures on a real finite-dimensional vector space.

Symplectic manifolds are introduced in the second chapter, and a symplectic homomorphism φ from a symplectic manifold (M_1, ω_1) to another symplectic manifold (M_2, ω_2) is defined. Contrary to the usual practice, it is not assumed that $\dim M_1 = \dim M_2$. Of course we must have $\dim M_1 \leq \dim M_2$, and rank $\varphi = \dim M_1$, but there are interesting cases in which $\dim M_1 < \dim M_2$. Several examples are discussed: quotients of $\mathbb{R}^{2n}$ by the action of a discrete subgroup of the group of translations, Kähler manifolds, the complex projective space $\mathbb{C}\mathrm{P}^n$. A long section, which uses results obtained in Chap. 1 about the canonical representation of $sl(2,k)$ in the graded vector space of skew-symmetric multilinear forms on V, is devoted to operators on the space of differential forms on a symplectic manifold, allowing the construction of a representation of the Lie algebra $sl(2,\mathbb{R})$. When there exists a 1-form α such that $\omega = \mathrm{d}\,\alpha$, it is proven that this representation can be extended to a representation of the Lie superalgebra $osp(2,\mathbb{R})$. Then *symplectic local coordinates* (often called, in other texts, *canonical coordinates* or *Darboux coordinates*) are defined, and a proof by induction of the Darboux theorem is given. In that proof, the rank of the form ω is assumed to be constant, but it may be strictly smaller than the dimension of the considered manifold. *Symplectic vector fields* on a symplectic manifold are then defined (in other texts, they are often called *locally Hamiltonian* vector fields), as well as *Hamiltonian vector fields*. Their properties are discussed. For example, it is proven that the vector space $S(M,\omega)$ of symplectic vector fields is a Lie subalgebra of the Lie algebra of all smooth vector fields, and that the Lie bracket of two symplectic vector fields is a Hamiltonian vector field. After a concise definition of de Rham cohomology groups, it is proven that the sequence

$$\{0\} \to H^0(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}) \to S(M,\omega) \to H^1(M,\mathbb{R}) \to \{0\}$$

is exact. Examples are given of vector fields which are symplectic but not Hamiltonian.

Next, the Poisson bracket of smooth functions on a symplectic manifold (M,ω) is defined. It is proven that $C^\infty(M,\mathbb{R})$, endowed with the Poisson bracket, is a Lie algebra, and that the map which associates to each smooth function the corresponding Hamiltonian vector field is a Lie algebra homomorphism whose kernel is the vector subspace $H^0(M)$ of locally constant functions.

Let $\varphi : P \to Q$ be a smooth map between two symplectic manifolds (P,ω_P) and (Q,ω_Q). For each smooth vector field Y defined on Q, there exists a unique vector field on P, denoted by φ^*Y, such that $\mathrm{i}(\varphi^*Y)\omega_P = \varphi^*(\mathrm{i}(Y)\omega_Q)$. By using this correspondence between vector fields when φ is a symplectic homomorphism, several orthogonality properties are established. It is proven that when φ is a symplectic homomorphism between symplectic manifolds of the same dimension, for any pair (f,g) of smooth functions defined on Q, $\varphi^*(\{f,g\}_Q = \{\varphi^*f,\varphi^*g\}_P$. This means that φ is a Poisson map. However, this result is not stated in these terms, because Poisson manifolds and Poisson maps (called in this book *Poisson homomorphisms*) are defined later, in Chap. 4. This result does not hold when φ is a symplectic homomorphism between two symplectic manifolds of different dimensions. In other words, symplectic homomorphisms between symplectic manifolds of different dimensions are not Poisson maps.

The set of polynomials defined on $\mathbb{R}^{2n}$ (endowed with its canonical symplectic structure) has both an associative algebra structure (for the ordinary product) and a Lie algebra structure (for the Poisson bracket). A careful study of its ideals is presented. Then the Lie algebra of formal symplectic vector fields defined on a symplectic manifold is discussed, and it is proven that the algebra of jets of symplectic vector fields at all points of a symplectic manifold is a simple algebra.

Isotropic, coisotropic, and Lagrangian submanifolds of a symplectic manifold are defined, and their main properties are established. The semi-global generalizations of the Darboux theorem, due to the American mathematician Alan Weinstein, are given. Then, a method for constructing Lagrangian submanifolds, or more generally Lagrangian immersions, called the *contraction method*, is presented. I believe that this method, called in other texts the use of *Morse families*, is due to the Swedish mathematician Lars Hörmander [F3 Hor].[1] At the end of the second chapter, an exercise is proposed in which it is proven that there exists, on each leaf of a Lagrangian foliation, a flat and torsionless connection. That result was observed as early as 1953 by the French mathematician Paulette Libermann in her thesis [F3 Lib].

The symplectic manifolds considered in Chap. 3 are cotangent bundles endowed with their canonical symplectic form. The Liouville 1-form on the total space of a cotangent bundle is first defined, and its main properties and its behavior under the prolongation of a diffeomorphism to covectors are presented. The canonical symplectic form is then defined as the *opposite* of the exterior differential of the Liouville form. A detailed discussion of symplectic vector fields on a cotangent bundle is given, in which the Liouville vector field C (the infinitesimal generator of homotheties in the fibers) is used. Functions which are homogeneous of a given degree with respect to the Lie derivation along C, in other words, functions whose restriction to each fiber of the cotangent bundle is a polynomial of a given degree $r+1$, and the associated Hamiltonian vector fields, are considered and a detailed study is made of the cases when $r=-1$, 0, and 1.

A deep study of Lagrangian submanifolds of a cotangent bundle is presented in the next section. It is first proven that the image of a 1-form is a Lagrangian submanifold if and only if that 1-form is closed. When it is the differential of a smooth function, that function is called a *generating function* for the corresponding Lagrangian submanifold. Then, following the beautiful booklet of Alan Weinstein [29], more general generating functions are introduced, now defined on an open subset of the cotangent bundle, whose associated Lagrangian submanifolds may not be transverse to the fibers. That construction is closely related to the *contraction method* described in Chap. 2.

Chapter 4 begins with the definition of actions of a Lie group and of a Lie algebra on a smooth manifold. A smooth manifold M on which a Lie group G acts smoothly on the left is called a *G-space*. An action of a finite-dimensional Lie algebra $\mathfrak{h}$ on a smooth manifold M is a Lie algebra homomorphism of $\mathfrak{h}$ in the Lie algebra of smooth vector fields on M (equipped with the Lie bracket as composition law). When the smooth manifold M is endowed with a symplectic form ω, a smooth

[1]Reference tags beginning with F3, such as [F3 Hor], point to the references list at the end of this Foreword, while reference tags such as [29] point to the main references list, at the end of this book.

action on M of a Lie group G is said to be *symplectic*, and (M, ω) is said to be a *symplectic G-space*, if the diffeomorphism of M associated to each element in G is a symplectic diffeomorphism. Similarly, an action on M of a finite-dimensional Lie algebra $\mathfrak{h}$ is said to be *symplectic*, and (M, ω) is said to be a *symplectic $\mathfrak{h}$-space*, when the vector field associated to each element in $\mathfrak{h}$ is a symplectic vector field. It is shown that the natural prolongation to the cotangent bundle T^*N of any smooth action of a Lie group on M is symplectic (later on it will be shown that such an action is Hamiltonian, after the definition of that notion). Other examples are presented. It is shown that each orbit of a symplectic action is an immersed submanifold on which ω induces a form of constant rank, and that the set of fixed points of a symplectic action of a compact Lie group on a symplectic manifold is a symplectic submanifold.

A symplectic action Γ of a Lie algebra $\mathfrak{g}$ on a symplectic manifold (M, ω) is said to be *Hamiltonian*, and (M, ω) is said to be a *Hamiltonian $\mathfrak{g}$-space*, when the vector field $\Gamma(a)$ associated to each $a \in \mathfrak{g}$ is a Hamiltonian vector field. A *moment map μ* of the Hamiltonian action Γ is defined: it is a smooth map μ from M to the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ such that, for each $a \in \mathfrak{g}$, the smooth function on M: $x \mapsto \langle \mu(x), a \rangle$ is a Hamiltonian for the Hamiltonian vector field $\Gamma(a)$. The properties of moment maps are established and it is shown that the difference of two moment maps for the same Hamiltonian action is a locally constant map. Moreover, for any moment map $\mu : M \to \mathfrak{g}$ and any pair (a, b) of elements in $\mathfrak{g}$, the function

$$x \mapsto c_\mu(a, b)(x) = \{\langle \mu(x), a \rangle, \langle \mu(x), b \rangle\} - \langle \mu(x), [a, b] \rangle$$

is locally constant on M. Therefore, c_μ is a skew-symmetric bilinear map on M which takes its values in $H^0(M)$. It satisfies a remarkable identity, a consequence of the Jacobi identity. Called a *closed* 2-*cochain* in this book, it is called a 2-*cocycle* of $\mathfrak{g}$ in other texts. Its restriction to each connected component of M is a skew-symmetric 2-form on $\mathfrak{g}$. When the moment map μ is modified by addition of a locally constant map φ taking its values in $\mathfrak{g}^*$, c_μ is modified by addition of the coboundary of φ. The cohomology class of c_μ remains unchanged: it depends only on the considered Hamiltonian $\mathfrak{g}$-space (M, ω), not on the choice of a moment map of the $\mathfrak{g}$-action. It is denoted by $\underline{c}(M, \omega)$. A Hamiltonian $\mathfrak{g}$-space (M, ω) is said to be *strongly Hamiltonian* when $\underline{c}(M, \omega) = \{0\}$. For reasons which will become clear in the next chapter, strongly Hamiltonian $\mathfrak{g}$-spaces are also called *Poisson $\mathfrak{g}$-spaces*.

Then it is proven that when the symplectic manifold (M, ω) is exact, *i.e.*, when there exists a 1-form α such that $\omega = -\,\mathrm{d}\,\alpha$, and when the considered symplectic action Γ of the Lie algebra $\mathfrak{g}$ is such that the Lie derivatives of α with respect to the vector fields in $\Gamma(\mathfrak{g})$ all vanish, that action is strongly Hamiltonian. Using the second Whitehead lemma (quoted without proof), it is shown that any symplectic action of a semi-simple Lie algebra is strongly Hamiltonian. Several examples are given of actions which are symplectic but not Hamiltonian, or Hamiltonian but not strongly Hamiltonian, and of strongly Hamiltonian actions. Some general properties of the moment map μ of a Hamiltonian action are proven. For example, it is shown that at each point x of a Hamiltonian $\mathfrak{g}$-space (M,ω), the kernel of the map $\mathrm{d}\,\mu_x$ (denoted by $T_x\mu$ in other texts) and $\{\Gamma_a(x)|a\in\mathfrak{g}\}$ are two symplectically orthogonal vector subspaces of the symplectic vector space $(T_xM,\omega(x))$. Moreover, the image of $T_x\mu$ in $\mathfrak{g}^*$ is the annihilator of the isotropy subalgebra $\mathfrak{g}_x$ of x (called in this book *the orthogonal complement of* $\mathfrak{g}_x$). When the Hamiltonian action of the Lie algebra $\mathfrak{g}$ comes from a Hamiltonian action of a Lie group G whose Lie algebra is $\mathfrak{g}$, several consequences are deduced from these properties, about the moment map and fixed points of the action.

The next section of Chap. 4 is devoted to equivariance properties of the moment map. Definitions of the adjoint and coadjoint actions of a Lie group G and of its Lie algebra $\mathfrak{g}$ are recalled. Then, it is shown that the moment map μ of a Hamiltonian action of a Lie group G on a connected Hamiltonian G-space (M,ω) is equivariant with respect to the considered action of G on M, and an affine action of that group on $\mathfrak{g}^*$, obtained by adding to the coadjoint action a closed cochain (in other words, a cocycle). When the considered action of G is strongly Hamiltonian, that cocycle is a coboundary; therefore, the addition of a suitably chosen constant to the moment map yields a moment map equivariant with respect to the coadjoint action. Two special cases are then considered: when the Lie group G is Abelian, and when its action on the manifold M is transitive. Readers are referred to the works of B. Kostant [17] and J.-M. Souriau [23] for more details.

Chapter 5 begins with a statement, without proof, of the main properties of the Schouten–Nijenhuis bracket. For each integer $p\geq 0$, the vector space of smooth p-multivectors on a smooth manifold M is denoted by $D_p(M)$, with the convention $D_0(M)=C^\infty(M,\mathbb{R})$. The direct sum of the vector spaces $D_p(M)$ for all $p\in\mathbb{N}$) is denoted by $D_*(M)$. The Schouten–Nijenhuis bracket is a bilinear, graded skew-symmetric composition law on $D_*(M)$ which maps $D_p(M)\times D_q(M)$ into $D_{p+q-1}(M)$. With the use of an element $w\in D_2(M)$, one can define a bilinear and skew-symmetric composition law $(f,g)\mapsto\{f,g\}_w=w(\mathrm{d}f,\mathrm{d}\,g)$ on $C^\infty(M,\mathbb{R})$ and a linear map $f\mapsto H_f$ from $C^\infty(M,\mathbb{R})$ in the vector space $D_1(M)$ of smooth vector fields on M. It is shown that the following four conditions are equivalent: (i) the composition law $(f,g)\mapsto\{f,g\}_w$ satisfies the Jacobi identity; (ii) for any pair (f,g) of elements in $C^\infty(M,\mathbb{R})$, $H_{\{f,g\}_w}=[H_f,H_g]$; (iii) for each $f\in C^\infty(M,\mathbb{R})$, $[H_f,w]=0$ (the bracket in the left-hand side being the Schouten–Nijenhuis bracket which, in this case, is the Lie derivative of w with respect to the vector field H_f); (iv) $[w,w]=0$ (the bracket in the left-hand side being the Schouten–Nijenhuis bracket). A *Poisson structure* on M is the structure determined on that manifold by a 2-multivector $w\in D_2(M)$ which satisfies these four equivalent conditions. The manifold M is then called a *Poisson manifold* and denoted by (M,w). *Poisson homomorphisms* (often called in other texts Poisson maps) between two Poisson manifolds are then defined.

It is explained in the second section of Chap. 5 that any Poisson manifold is a union of immersed symplectic manifolds, called its *symplectic leaves*. The proof of that property, rather delicate, is not fully given; it is only observed that the result is true when the rank of the bivector field w which determines the Poisson structure is constant, and asserted that the same is true in the general case. It is proven that each integral manifold of the family of vector fields $\{H_f \,|\, f \in C^\infty(M, \mathbb{R})\}$ is endowed with a symplectic form such that, for each point $x \in M$ and each pair (f, g) of smooth functions defined on an open neighborhood of x, the following property is satisfied: $\{f, g\}_w(x)$, defined as the value at x of the evaluation $w(\mathrm{d}f, \mathrm{d}g)$ of w on the two 1-forms $\mathrm{d}f$ and $\mathrm{d}g$, is equal to the value at x of the Poisson bracket of the restrictions of f and g to the integral manifold of $\{H_f \,|\, f \in C^\infty(M, \mathbb{R}R)\}$ which contains x, calculated with the use of its symplectic form as indicated in Chap. 2. A Poisson manifold which has only one symplectic leaf is a symplectic manifold, the bivector field w being, in a certain sense, the inverse of its symplectic form. A simple example is given at the end of that section.

Poisson structures on the dual vector space of a finite-dimensional Lie algebra are studied in detail in the last section of Chap. 5. The bracket of a finite-dimensional Lie algebra $\mathfrak{g}$, considered as a composition law of linear functions defined on the dual space $\mathfrak{g}^*$, is first extended to a composition law on $C^\infty(\mathfrak{g}^*, \mathbb{R})$. The Poisson structure so obtained on $\mathfrak{g}^*$ was observed by Sophus Lie and rediscovered, much later, independently by A. Kirillov, B. Kostant, and J.-M. Souriau. It is often called the *canonical Lie–Poisson structure* or the *Kirillov–Kostant–Souriau structure* on $\mathfrak{g}^*$. By observing that the transpose of a Lie algebra homomorphism is a Poisson homomorphism, it is proven that when $\mathfrak{g}$ is the Lie algebra of a Lie group G, the canonical Lie–Poisson structure on $\mathfrak{g}^*$ is preserved by the coadjoint action of G, and that the symplectic leaves of $\mathfrak{g}^*$ are the orbits of the coadjoint action, restricted to the neutral component of G. Restricted to each orbit, the action of G is Hamiltonian and has the canonical injection of that orbit in $\mathfrak{g}^*$ as moment map. It is then proven that the Ad^*-equivariant moment map of a strongly Hamiltonian action is a Poisson homomorphism.

The remainder of that section extends these results to Hamiltonian actions, which may not be strongly Hamiltonian, of a Lie group G on a connected symplectic manifold (M, ω). For any moment map μ of that action, it was proven in Chap. 4 that there exists an affine action of G on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ with respect to which μ is equivariant. That action is obtained by adding to the coadjoint action, which is its linear part, a smooth map $\varphi_\mu : G \to \mathfrak{g}^*$ which is a 1-cocycle of G with values in $\mathfrak{g}^*$ for the coadjoint action. It is now proven that the canonical Lie–Poisson structure on $\mathfrak{g}^*$ can be modified, by addition of the 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g}^*$ associated to φ_μ, in such a way that for this modified Poisson structure, the moment map μ becomes a Poisson homomorphism. This remarkable result was obtained for the first time, I believe, by J.-M. Souriau, and appears in his book [23]. The proof presented here is slightly different, and more algebraic. Moreover, it is proven here that when the Lie group G is simply connected, the 1-cocycle φ_μ of G and the associated 1-cocycle of its Lie algebra for which μ is an equivariant Poisson homomorphism can be deduced from the 2-cocycle of $\mathfrak{g}$ introduced in Chap. 4.

Chapter 5 ends with proposed exercises related to Lie groups and Lie algebras endowed with a symplectic structure, in which reference to the works of Vinberg is made [25].

Finally, Chap. 6, the most original part of this book, is a short introduction to supermanifolds, in particular, symplectic supermanifolds.

Some Developments in Symplectic and Poisson Geometry

Since 1983, several textbooks have appeared in the fields of symplectic and Poisson geometry, for example [F3 Cann, F3 Fo, F3 Lib-Ma], in which the readers will find other viewpoints on the subjects dealt with in this book. Readers interested in more specialized and advanced results are referred to the proceedings of conferences, for example [F3 Do, F3 Gra-Urb, F3 Mars-Ra]. Methods in symplectic geometry are used for the study of global geometric properties of completely integrable Hamiltonian systems in the book by M. Audin, A. Cannas da Silva, and E. Lerman [F3 Au-Can-Ler] and in the book by R. Cushman and L. M. Bates [F3 Cush-Ba] for classical mechanical systems.

Local properties of Poisson manifolds were thoroughly studied by A. Weinstein [F3 W1]. There exists on a Poisson manifold a cohomology discovered by A. Lichnerowicz, defined by means of the Poisson bivector field, called the *Lichnerowicz–Poisson cohomology*. When the considered Poisson manifold is in fact a symplectic manifold, its Lichnerowicz–Poisson cohomology coincides with its de Rham cohomology. For general Poisson manifolds, the Lichnerowicz–Poisson cohomology is often more complicated than the de Rham cohomology because it reflects the topological and geometrical properties of the foliation of that manifold in symplectic leaves. It was studied by P. Xu [F3 Xu] and many other authors.

Remarkable properties of moment maps, not discussed in this book, have been discovered since 1982. The readers are referred to the important papers of M. Atiyah [F3 Ati], V. Guillemin and S. Sternberg [F3 Gu-Ster1, F3 Gu-Ster2], F. Kirwan [F3 Kir], T. Delzant [F3 Del], and to the beautiful book of M. Audin [F3 Au]. A deep study of moment maps and their use for reduction can be found in the monumental book by J.-P. Ortega and T. Ratiu [F3 Or-Ra].

Following the pioneering work of F. Magri [F3 Mag, F3 Ma-Mo], vector fields defined on a smooth manifold which are Hamiltonian with respect to two different symplectic or Poisson structures, with two different Hamiltonian functions, called *bi-Hamiltonian systems*, were intensively studied by many authors. Very often these systems are integrable and also occur in the more general setting of evolution partial differential equations.

Several important new results were found in the geometry of Poisson manifolds. Jean-Louis Koszul discovered the composition law on the graded vector space of differential forms of all degrees on a Poisson manifold [F3 Kosz]. Its restriction to the vector subspace of differential forms of degree 1 had already been observed a little earlier by F. Magri and C. Morosi [F3 Ma-Mo], and a little later by P. Dazord and D. Sondaz [F3 Da-Son]. It was then found by A. Weinstein, P. Dazord,

D. Sondaz, and probably many others [F3 W2, F3 W3, F3 Da-W, F3 Da-Son] that the tangent space to a Poisson manifold is endowed with a Lie algebroid structure. J.-P. Dufour and N. T. Zung's book [F3 Du-Zu] very thoroughly presents normal forms of Poisson structures. The book by C. Laurent-Gengoux, A. Pichereau, and P. Vanhaecke is an invaluable reference work about Poisson structures and their applications.

A *Poisson Lie group* is a Lie group G endowed with a Poisson structure for which the group composition law is a Poisson homomorphism from $G \times G$ equipped with the product Poisson structure, to G. Poisson Lie groups were introduced by V. G. Drinfel'd [F3 Dri] as classical analogues of quantum groups. The Lie algebra of a Poisson–Lie group has a remarkable property and is often called a *Lie bialgebra*: its dual vector space too is endowed with a Lie algebra structure. Poisson–Lie groups have been extensively studied by many authors, in particular, by J.-H. Lu and A. Weinstein [F3 Lu, F3 Lu-W].

Research on the symplectization of Poisson manifolds (the determination of a symplectic manifold of which the considered Poisson manifold is a quotient) and on the integration of Lie algebroids led A. Weinstein, S. Zakzrewski, M. Karasev, and others to the consideration of Lie groupoids in relation with Poisson manifolds and the definition of symplectic groupoids [F3 W2, F3 W3, F3 W4, F3 W5]. The readers are referred to the book by K.C.H. Mackenzie [F3 Mack] for a thorough study of Lie groupoids and to the booklet by A. Cannas da Silva and A. Weinstein [F3 Cann-W] for symplectic groupoids, the links of Lie groupoids with Poisson geometry, and more references.

Poisson groupoids, *i.e.*, Lie groupoids whose total space is endowed with a suitable Poisson structure, generalize both Poisson Lie groups and symplectic groupoids. Hamiltonian actions of a Poisson groupoid on a symplectic or Poisson manifold and moment maps for these actions can be defined, which generalize Hamiltonian actions and moment maps for a Lie group action [F3 Mack-Xu, F3 Gra-Urb, F3 Mars-Ra].

Symplectic topology is a development of symplectic geometry which appeared toward the end of the last century, in which symplectic geometric methods are used for the proof of topological properties of symplectic or contact manifolds, and for the definition of topological invariants. That development stemmed from the seminal work of M. Gromov [F3 Grom] and is now a very active domain of research. Readers interested in that subject are referred to the books by D. McDuff and D. Salamon [F3 McD-Sa] and by L. Polterovich and D. Rosen [F3 Pol-Ros] and the proceedings of an international conference [F3 Eli-Tray].

Meudon, France Charles-Michel Marle
May 2018

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Preface

I was invited to give lectures at Nankai University in the spring of 1983. This book is based on the lecture notes, translated and written (with minor modifications) by Yi Ming Zou. We hope to introduce symplectic manifold theory to the readers through this introductory book.

The development of analytical mechanics provided the basic concepts of symplectic structures. The term symplectic structure is due largely to analytical mechanics. But in this book, the applications of symplectic structure theory to mechanics are not discussed in any detail; and some of the important parts of the theory, especially the application in analysis, are not discussed at all. For those topics, we refer the readers to the references [1, 2, 7, 26]. The emphasis of this book is on the differential properties of manifolds with symplectic structures.

The first chapter of the book discusses the symplectic structures of vector spaces. The second chapter discusses symplectic manifolds and introduces the basic concepts and the basic results to the readers. We prove the existence of symplectic coordinates (Darboux Theorem) as early as possible in Chap. 2, so the readers can see the importance of the formulas we give in later discussions. The connection between the differentiable functions and the infinitesimal automorphisms of the symplectic structure on a symplectic manifold is the foundation of the symplectic manifold theory, and it will be discussed in Sects. 2.4 and 2.5. This chapter ends with some results on the submanifolds, especially the Lagrangian submanifolds, of a symplectic manifold.

The existence of canonical symplectic structures on cotangent bundles clarifies a lot of questions associated with symplectic structures. Chapter 3 introduces the results on cotangent bundles and symplectic vector fields on cotangent bundles.

Chapter 4 discusses symplectic G-spaces, that is, symplectic manifolds with symplectic structures that are invariant under the actions of some Lie group G. For these symplectic manifolds, certain maps we call moment maps provide with us an effective study tool. The study of symplectic G-spaces is a rich topic in symplectic manifold theory, and there are still many problems that deserve further study. The study of the symplectic G-spaces leads us to the study of the dual structures of Lie algebras and the geometry properties of the so-called coadjoint representations. We will discuss these subjects in Sect. 4.3, which will last until Chap. 5. In Chap. 5, we first introduce some general properties of the so-called Poisson manifolds. The concept of Poisson structures is a generalization of the concept of symplectic structures, and it allows us to consider the classical contents from a new viewpoint. Poisson structures start from the concept of contravariant skew-symmetric tensors. In Sect. 5.3, we will give the precise results for the Poisson structures in the dual spaces of Lie algebras.

Chapter 6, the last chapter, is a special chapter. The purpose of this chapter is to introduce the generalizations of the concepts discussed in Chaps. 2 and 3 to supermanifolds. We only discuss $(0, n)$-dimensional supermanifolds, that is, we only consider differentiable properties, not geometric properties. In this chapter, we mainly describe the basic properties, omit most the proofs. This is because the materials discussed are rather basic; and, we think that these omitted proofs can serve as exercises for the readers.

Finally, we wish to express our sincere thanks to Professor Zhi-da Yan for his help in the process of the translation and writing of this book.

Grenoble, France
December 1984

Jean-Louis Koszul

Notations

$\mathbb{Z}$	The ring of integers
$\mathbb{Z}^+$	The set of nonnegative integers
$\mathbb{R}$	Real number field
$\mathbb{C}$	Complex number field
$\mathbb{Z}_2$	The ring of integers modulo 2
$A^p(V)$	Skew-symmetric p-linear forms on V
(V, ω)	Symplectic vector space
$\underline{\omega} = \sum_{i=1}^{r} f_i \wedge f_{r+i}$	Canonical symplectic form on k^{2r}
$Sp(V, \omega)$	Symplectic group
$sp(V, \omega)$	Symplectic Lie algebra
$O(V, b)$	Orthogonal transformation group of V w.r.t bilinear form b
$O(2n)$	Orthogonal group
$U(n)$	Unitary group
TM	Tangent bundle of M
T^*M	Cotangent bundle of M
$T(T^*P)$	Tangent bundle of the cotangent bundle T^*P
$T^*(X)$	Extension of the vector field X
$C^\infty(M)$	Real C^∞-differentiable functions on M
$\Omega^p(M)$	Differential p-forms on M
$D_p(M)$	Degree p skew symm. contravariant diff. tensor fields on M
$\mathrm{i}(X)$	The inner product through X
$\theta(X)$	The Lie derivation defined by X
$\mathbb{C}P^n$	Complex projective space
$\mu(\alpha)$	Left exterior product defined by α
$osp(2, 1)$	5-dimensional orthosymplectic Lie superalgebra
(M, ω)	Symplectic manifold
$S(M, \omega)$	Symplectic vector fields on (M, ω)
$Z^p(M)$	Space of all closed p-forms on M
$H^p(M) = H^p(M, \mathbb{R})$	p-dimensional de Rham group of M
H_f	The vector field on M defined by $f \in C^\infty(M)$

$\{f,g\}$	Poisson bracket of f,g
$J_x(M)$	Jet algebra at the point $x \in M$
$J_x(X)$	Jet of the vector field X at the point x
$\mathfrak{g}$	Lie algebra
$G(x)$	G-orbit of the point x
$\mathfrak{g}x$	Tangent space of $G(x)$ at x
G_x	Isotropy subgroup of G at the point x
$\mathfrak{g}_x$	Lie algebra of G_x
Ad	Adjoint representation of G on $\mathfrak{g}$
ad	Adjoint representation of $\mathfrak{g}$ on itself
Ad^*	Coadjoint representation of G on $\mathfrak{g}^*$
$[\ ,\ ]_S$	Schouten–Nijenhuis bracket of $D_*(\mathfrak{g}^*)$
$\Lambda(k^n)$	Rank n Grassmann algebra over k
$\to$	Map between sets
$\mapsto$	Map between elements

Contents