Lie Group Machine Learning & Gibbs Density on Poincaré Unit Disk from Souriau Lie Groups Thermodynamics and SU(1,1) Coadjoint Orbits

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Abstract. In 1969, Jean-Marie Souriau has introduced a "Lie Groups Thermodynamics" in Statistical Mechanics in the framework of Geometric Mechanics. This Souriau's model considers the statistical mechanics of dynamic systems in their "space of evolution" associated to a homogeneous symplectic manifold by a Lagrange 2-form, and defines thanks to cohomology (non equivariance of the coadjoint action on the moment map with appearance of an additional cocyle) a Gibbs density (of maximum entropy) that is covariant under the action of dynamic groups of physics (eg, Galileo's group in classical physics). Souriau model is more general if we consider another Souriau theorem, that we can associate to a Lie group, an homogeneous symplectic manifold with a KKS 2-form on their coadjoint orbits. Souriau method could then be applied on Lie Groups to define a covariant maximum entropy density by Kirillov representation theory. We will illustrate this method for homogeneous Siegel domains and more especially for Poincaré unit disk by considering SU(1,1) group coadjoint orbit and by using its Souriau's moment map. For this case, the coadjoint action on moment map is equivariant.

Keywords: Lie Groups Thermodynamics, Lie Group Machine Learning, Kirillov Representation Theory, Coadjoint Orbits, Moment Map, Covariant Gibbs Density, Maximum Entropy Density, Souriau-Fisher Metric.

1 Lie Groups Thermodynamics and Covariant Gibbs Density

We identify the Riemanian metric introduced by Souriau based on cohomology, in the framework of "Lie groups thermodynamics" as an extension of classical Fisher metric introduced in information geometry. We have observed that Souriau metric preserves Fisher metric structure as the Hessian of the minus logarithm of a partition function, where the partition function is defined as a generalized Laplace transform on a sharp convex cone. Souriau's definition of Fisher metric extends the classical one in case of Lie groups or homogeneous manifolds. Souriau has developed this "Lie groups thermodynamics" theory in the framework of homogeneous symplectic manifolds in geometric statistical mechanics for dynamical systems, but as observed by Souriau, these model equations are no longer linked to the symplectic manifold but equations only depend on the Lie group and the associated cocycle. This analogy with Fisher metric opens potential applications in machine learning, where the Fisher metric is used in the framework of information geometry, to define the "natural gradient" tool for improving ordinary stochastic gradient descent sensitivity to rescaling or changes of variable in parameter space. In machine learning revised by natural gradient of information geometry, the ordinary gradient is designed to integrate the Fisher matrix. Amari has theoretically proved the asymptotic optimality of the natural gradient compared to classical gradient. With the Souriau approach, the Fisher metric could be extended, by Souriau-Fisher metric, to design natural gradients for data on homogeneous manifolds. Information geometry has been derived from invariant geometrical structure involved in statistical inference. The Fisher metric defines a Riemannian metric as the Hessian of two dual potential functions, linked to dually coupled affine connections in a manifold of probability distributions. With the Souriau model, this structure is extended preserving the Legendre transform between two dual potential function parametrized in Lie algebra of the group acting transentively on the homogeneous manifold. Classically, to optimize the parameter θ of a probabilistic model, based on a sequence of observations y_t, is an online gradient descent:

$$\theta_t \leftarrow \theta_{t-1} - \eta_t \frac{\partial l_t(y_t)^T}{\partial \theta} \tag{1}$$

with learning rate η_t, and the loss function $l_t = -\log p(y_t / \hat{y}_t)$. This simple gradient descent has a first drawback of using the same non-adaptive learning rate for all parameter components, and a second drawback of non invariance with respect to parameter re-encoding inducing different learning rates. Amari has introduced the natural gradient to preserve this invariance to be insensitive to the characteristic scale of each parameter direction. The gradient descent could be corrected by $I(\theta)^{-1}$ where I is the Fisher information matrix with respect to parameter θ, given by:

$$I(\theta) = \left[g_{ij}\right] \text{ with } g_{ij} = \left[-E_{y\sim p(y/\theta)}\left[\frac{\partial^2 \log p(y/\theta)}{\partial\theta_i\partial\theta_j}\right]\right]_{ij} \tag{2}$$

with natural gradient: $$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta)^{-1} \frac{\partial l_t(y_t)^T}{\partial \theta} \tag{3}$$

Amari has proved that the Riemannian metric in an exponential family is the Fisher information matrix defined by:

$$g_{ij} = -\left[\frac{\partial^2 \Phi}{\partial\theta_i\partial\theta_j}\right]_{ij} \text{ with } \Phi(\theta) = -\log \int_{\mathbb{R}} e^{-\langle\theta, y\rangle} dy \tag{4}$$

and the dual potential, the Shannon entropy, is given by the Legendre transform:

$$S(\eta) = \langle\theta, \eta\rangle - \Phi(\theta) \text{ with } \eta_i = \frac{\partial\Phi(\theta)}{\partial\theta_i} \text{ and } \theta_i = \frac{\partial S(\eta)}{\partial\eta_i} \tag{5}$$

In geometric statistical mechanics, Souriau has developed a "Lie groups thermodynamics" of dynamical systems where the (maximum entropy) Gibbs density is covariant with respect to the action of the Lie group. In the Souriau model, previous structures of information geometry are preserved:

$$I(\beta)=-\frac{\partial^2\Phi}{\partial\beta^2} \text{ with } \Phi(\beta)=-\log\int_M e^{-\langle\beta,U(\xi)\rangle}d\omega \text{ and } U:M\to\mathfrak{g}^* \tag{6}$$

$$S(Q)=\langle\beta,Q\rangle-\Phi(\beta) \text{ with } Q=\frac{\partial\Phi(\beta)}{\partial\beta}\in\mathfrak{g}^* \text{ and } \beta=\frac{\partial S(Q)}{\partial Q}\in\mathfrak{g} \tag{7}$$

In the Souriau Lie groups thermodynamics model, β is a "geometric" (Planck) temperature, element of Lie algebra $\mathfrak{g}$ of the group, and Q is a "geometric" heat, element of dual Lie algebra $\mathfrak{g}^*$ of the group. Souriau has proposed a Riemannian metric that we have identified as a generalization of the Fisher metric:

$$I(\beta)=\left[g_\beta\right] \text{ with } g_\beta\left([\beta,Z_1],[\beta,Z_2]\right)=\tilde{\Theta}_\beta\left(Z_1,[\beta,Z_2]\right) \tag{8}$$

$$\text{with } \tilde{\Theta}_\beta(Z_1,Z_2)=\tilde{\Theta}(Z_1,Z_2)+\left\langle Q,ad_{Z_1}(Z_2)\right\rangle \text{ where } ad_{Z_1}(Z_2)=[Z_1,Z_2] \tag{9}$$

Souriau has proved that all co-adjoint orbit of a Lie Group given by $\mathrm{O}_F=\left\{Ad_g^*F=g^{-1}Fg, g\in G\right\}$ subset of $\mathfrak{g}^*, F\in\mathfrak{g}^*$ carries a natural homogeneous symplectic structure by a closed G-invariant 2-form. If we define $K=Ad_g^*=\left(Ad_{g^{-1}}\right)^*$ $K_*(X)=-\left(ad_X\right)^*$ with $\left\langle Ad_g^*F,Y\right\rangle=\left\langle F,Ad_{g^{-1}}Y\right\rangle,\forall g\in G,Y\in\mathfrak{g},F\in\mathfrak{g}^*$ where if $X\in\mathfrak{g}$, $Ad_g(X)=gXg^{-1}\in\mathfrak{g}$, the G-invariant 2-form is given by the following expression $\sigma_\Omega\left(K_{*X}F,K_{*Y}F\right)=B_F(X,Y)=\left\langle F,[X,Y]\right\rangle, X,Y\in\mathfrak{g}$. Souriau Foundamental Theorem is that « every symplectic manifold is a coadjoint orbit ». We can observe that for Souriau model (8), Fisher metric is an extension of this 2-form in non-equivariant case $g_\beta\left([\beta,Z_1],[\beta,Z_2]\right)=\tilde{\Theta}\left(Z_1,[\beta,Z_2]\right)+\left\langle Q,\left[Z_1,[\beta,Z_2]\right]\right\rangle$.
The Souriau additional term $\tilde{\Theta}\left(Z_1,[\beta,Z_2]\right)$ is generated by non-equivariance through Symplectic cocycle. The tensor $\tilde{\Theta}$ used to define this extended Fisher metric is defined by the moment map $J(x)$, application from M (homogeneous symplectic manifold) to the dual Lie algebra $\mathfrak{g}^*$, given by: $\tilde{\Theta}(X,Y)=J_{[X,Y]}-\left\{J_X,J_Y\right\}$ (10)

$$\text{with } J(x):M\to\mathfrak{g}^* \text{ such that } J_X(x)=\langle J(x),X\rangle, X\in\mathfrak{g} \tag{11}$$

This tensor $\tilde{\Theta}$ is also defined in tangent space of the cocycle $\theta(g)\in\mathfrak{g}^*$ (this cocycle appears due to the non-equivariance of the coadjoint operator Ad_g^*, action of the group on the dual lie algebra): $Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta(g)$ (12)

$$\begin{aligned}\tilde{\Theta}(X,Y):\ &\mathfrak{g}\times\mathfrak{g}\to\Re \qquad \text{with } \Theta(X)=T_e\theta\left(X(e)\right)\\ &X,Y\mapsto\langle\Theta(X),Y\rangle\end{aligned} \tag{13}$$

In Souriau's Lie groups thermodynamics, the invariance by re-parameterization in information geometry has been replaced by invariance with respect to the action of the group. When an element of the group g acts on the element $\beta \in \mathfrak{g}$ of the Lie algebra, given by adjoint operator Ad_g. Under the action of the group $Ad_g(\beta)$, the entropy $S(Q)$ and the Fisher metric $I(\beta)$ are invariant:

$$\beta \in \mathfrak{g} \to Ad_g(\beta) \Rightarrow \begin{cases} S\left[Q\left(Ad_g(\beta)\right)\right] = S(Q) \\ I\left[Ad_g(\beta)\right] = I(\beta) \end{cases} \tag{14}$$

In the framework of Lie group action on a symplectic manifold, equivariance of moment could be studied to prove that there is a unique action $a(.,.)$ of the Lie group G on the dual $\mathfrak{g}^*$ of its Lie algebra for which the moment map J is equivariant, that means for each $x \in M$: $J\left(\Phi_g(x)\right) = a(g, J(x)) = Ad_g^*\left(J(x)\right) + \theta(g)$ (15)

When coadjoint action is not equivariant, the symmetry is broken, and new "cohomological" relations should be verified in Lie algebra of the group. A natural equilibrium state will thus be characterized by an element of the Lie algebra of the Lie group, determining the equilibrium temperature β. The entropy $s(Q)$, parametrized by Q the geometric heat (mean of energy U, element of the dual Lie algebra) is defined by the Legendre transform of the Massieu potential $\Phi(\beta)$ parametrized by β ($\Phi(\beta)$ is the minus logarithm of the partition function $\psi_\Omega(\beta)$). Souriau has then defined a Gibbs density that is covariant under the action of the group:

$$\begin{gathered} p_{Gibbs}(\xi) = e^{\Phi(\beta) - \langle \beta, U(\xi) \rangle} = \frac{e^{-\langle \beta, U(\xi) \rangle}}{\int\limits_M e^{-\langle \beta, U(\xi) \rangle} d\omega} \quad \text{, with } \Phi(\beta) = -\log \int\limits_M e^{-\langle \beta, U(\xi) \rangle} d\omega \\ Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \frac{\int\limits_M U(\xi) e^{-\langle \beta, U(\xi) \rangle} d\omega}{\int\limits_M e^{-\langle \beta, U(\xi) \rangle} d\omega} = \int\limits_M U(\xi) p(\xi) d\omega \end{gathered} \tag{16}$$

We will illustrate computation of this covariant Souriau-Gibbs density for the Lie group *SU(1,1)* and the unit disk considered as an homogeneous symplectic manifold.

2 Souriau Moment map

$i_V\omega$ is the (p-1)-form on M obtained by inserting $V(x)$ as the first argument of ω:

Interior product $i_V\omega\left(v_2, \cdots v_p\right) = \omega\left(V(x), v_2, \cdots, v_p\right)$ (17)

$\theta \wedge \omega$ is the (p + 1)-form on X where ω is a p-form and θ is a 1-form on M:

Exterior product $\theta \wedge \omega\left(v_0, \cdots, v_p\right) = \sum_{i=0}^{p} (-1)^i \theta(v_i) \omega\left(v_0, \cdots, \hat{v}_i, \cdots, v_p\right)$ (where the hat indicates a term to be omitted).

$L_V\omega$ is a p-form on M, and $L_V\omega = 0$ if the flow of V consists of symmetries of ω:

Lie derivative $L_V\omega\left(v_1,\cdots,v_p\right)=\frac{d}{dt}e^{tV*}\omega\left(v_1,\cdots,v_p\right)\Big|_{t=0}$ (18)

$d\omega$ is the (p+1)-form on M defined by taking the ordinary derivative of ω and then antisymmetrizing:

Exterior derivative $d\omega\left(v_0,\cdots,v_p\right)=\sum_{i=0}^{p}\left(-1\right)^i\frac{\partial\omega}{\partial x}(v_i)(v_0,\cdots,\hat{v}_i,\cdots,v_p)$ (19)

$p=0,\left[d\omega\right]_i=\partial_i\omega$; $p=1,\left[d\omega\right]_{ij}=\partial_i\omega_j-\partial_j\omega_i$; $p=2,\left[d\omega\right]_{ijk}=\partial_i\omega_{jk}+\partial_j\omega_{ki}+\partial_k\omega_{ij}$

The properties of the exterior and Lie Derivative are the following:

$L_V\omega=di_V\omega+i_Vd\omega$ (***E. Cartan***), $i_{[U,V]}\omega=i_VL_U\omega-L_Ui_V\omega$ (***H.Cartan***) (20)

$L_{[U,V]}\omega=L_VL_U\omega-L_UL_V\omega$ (***S. Lie***) (21)

Let $\left(M,\sigma\right)$ be a connected symplectic manifold. A vector field η on M is called symplectic if its flow preserves the 2-form : $L_\eta\sigma=0$. If we use Elie Cartan's formula, we can deduce that $L_\eta\sigma=di_\eta\sigma+i_\eta d\sigma=0$ but as $d\sigma=0$ then $di_\eta\sigma=0$. We observe that the 1-form $i_\eta\sigma$ is closed. When this 1-form is exact, there is a smooth function $x\mapsto H$ on M with: $i_\eta\sigma=-dH$. This vector field η is called Hamiltonian and could be defined as symplectic gradient $\eta=\nabla_{Symp}H$.

Let a Lie group G that acts on M and that also preserve σ. A moment map exists if these infinitesimal generators are actually hamiltonian, so that a map $J:M\to\mathfrak{g}^*$ exists with $i_{Z_X}\sigma=-dH_Z$ where $H_Z=\left\langle J(x),Z\right\rangle$ (22)

We define also the Poisson bracket of two functions H, H' by :

$\left\{H,H'\right\}=\sigma\left(\eta,\eta'\right)=\sigma\left(\nabla_{Symp}H',\nabla_{Symp}H\right)$ with $i_\eta\sigma=-dH$ and $i_{\eta'}\sigma=-dH'$ (23)

3 Coadjoint orbits and Moment Map for SU(1,1)

3.1 Poincaré Unit Disk and SU(1,1) Lie Group

The group of complex unimodular pseudo-unitary matrices $SU(1,1)$, is the set of elements u such that: $uMu^+=M$ with $M=\begin{pmatrix}+1 & 0\\ 0 & -1\end{pmatrix}$ (22)

We can show that the most general matrix u belongs to the Lie group given by:

$$G=SU(1,1)=\left\{\begin{pmatrix}a & b\\ b^* & a^*\end{pmatrix}/\left|a\right|^2-\left|b\right|^2=1,\ a,b\in\mathbb{C}\right\} \quad (23)$$

Its Cartan decomposition is given by:

$$\begin{pmatrix}a & b\\ b^* & a^*\end{pmatrix}=\left|a\right|\begin{pmatrix}1 & z\\ z^* & 1\end{pmatrix}\begin{pmatrix}a/\left|a\right| & 0\\ 0 & a^*/\left|a\right|\end{pmatrix} \text{ with } z=b\left(a^*\right)^{-1},\left|a\right|=\left(1-\left|z\right|^2\right)^{-1/2} \quad (24)$$

$$\begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}\begin{pmatrix} 1 & z \\ z^* & 1 \end{pmatrix} = |a'| \begin{pmatrix} 1 & z' \\ z'^* & 1 \end{pmatrix}\begin{pmatrix} a'/|a'| & 0 \\ 0 & a'^*/|a'| \end{pmatrix} \text{ with } \begin{cases} a' = bz^* + a \\ z' = \dfrac{az+b}{b^* z + a^*} \end{cases} \tag{25}$$

$SU(1,1)$ is associated to group of holomorphic automorphisms of the Poincaré unit disk $D = \{ z = x + iy \in \mathbb{C} / |z| < 1 \}$ in the complex plane, by considering its action on the disk as $g(z) = (az+b)/(b^* z + a^*)$. The following measure on Unit disk:

$$d\mu_0(z, z^*) = \frac{1}{2\pi i} \frac{dz \wedge dz^*}{\left(1 - |z|^2\right)^2} \tag{26}$$

is invariant under the action of $SU(1,1)$ captured by the fractional holomorphic transformation: $\dfrac{dz' \wedge dz'^*}{\left(1-|z'|^2\right)^2} = \dfrac{dz \wedge dz^*}{\left(1-|z|^2\right)^2}$ (27)

The complex unit disk admits a Kähler structure determined by potential function:

$$\Phi(z', z^*) = -\log(1 - z' z^*) \tag{28}$$

The invariant 2-form is: $\Omega = \dfrac{1}{i} \dfrac{\partial^2 \Phi(z, z^*)}{\partial z \partial z^*} dz \wedge dz^* = \dfrac{1}{i} \dfrac{dz \wedge dz^*}{\left(1-|z|^2\right)^2}$ (29)

which is closed $d\Omega = 0$. This group $SU(1,1)$ is isomorphic to the group $SL(2,\mathbb{R})$ as a real Lie group, and the Lie algebra $\mathfrak{g} = \mathfrak{su}(1,1)$ is given by:

$$g = \left\{ \begin{pmatrix} -ir & \eta \\ \eta^* & ir \end{pmatrix} / r \in \mathbb{R}, \eta \in \mathbb{C} \right\} \tag{30}$$

with the bases $(u_1, u_2, u_3) \in \mathfrak{g}$: $u_1 = \dfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $u_2 = \dfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $u_3 = \dfrac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

with the commutation relation: $[u_3, u_2] = u_1, [u_3, u_1] = -u_2, [u_2, u_1] = -u_3$ (31)

Dual base on dual Lie algebra is named $(u_1^*, u_2^*, u_3^*) \in \mathfrak{g}^*$. The dual vector space $\mathfrak{g}^* = \mathfrak{su}^*(1,1)$ can be identified with the subspace of $\mathfrak{sl}(2,\mathbb{C})$ of the form:

$$g^* = \left\{ \begin{pmatrix} z & x+iy \\ -x+iy & -z \end{pmatrix} = x \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / x, y, z \in \mathbb{R} \right\} \tag{32}$$

Coadjoint action of $g \in G$ on dual Lie algebra $\xi \in \mathfrak{g}^*$ is written $g.\xi$.

3.2 Coadjoint Orbit of SU(1,1) and Souriau Moment Map

We will use results of C. Cishahayo and S. de Bièvre [7] and B. Cahen [8,9] for computation of moment map of $SU(1,1)$. Let $r \in \mathbb{R}^{*+}$, orbit $\mathrm{O}(ru_3^*)$ of ru_3^* for the coadjoint action of $g \in G$ could be identified with the upper half sheet $x_3 > 0$ of $\left\{ \xi = x_1 u_1^* + x_2 u_2^* + x_3 u_3^* / -x_1^2 - x_2^2 + x_3^2 = r^2 \right\}$, the two-sheet hyperboloid. The stabilizer of ru_3^* for the coadjoint action of G is torus $K=\left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}, \theta\in\mathbb{R}\right\}$. K induces rotations of the unit disk, and leaves 0 invariant. The stabilizer for the origin 0 of unit disk is maximal compact subgroup K of *SU(1,1)*. We can observe [8] that $\mathrm{O}\left(ru_3^*\right)\simeq G/K$. On the other hand $\mathrm{O}\left(ru_3^*\right)\simeq G/K$ is diffeomorphic to the unit disk $D=\left\{z\in\mathbb{C}/\left|z\right|<1\right\}$, then by composition, the moment map is given by:

$$J: D\to \mathrm{O}\left(ru_3^*\right)$$

$$z\mapsto J(z)=r\left(\frac{z+z^*}{\left(1-|z|^2\right)}u_1^*+\frac{z-z^*}{i\left(1-|z|^2\right)}u_2^*+\frac{1+|z|^2}{\left(1-|z|^2\right)}u_3^*\right) \tag{33}$$

J is linked to the natural action of G on D (by fractional linear transforms) but also the coadjoint action of G on $\mathrm{O}\left(ru_3^*\right)\simeq G/K$. J^{-1} could be interpreted as the stereographic projection from the two-sphere S^2 onto $\mathbb{C}\cup\infty$. In case $r=\frac{n}{2}$ where $n\in\mathbb{N}^+, n\geq 2$ then the coadjoint orbit is given by $\mathrm{O}_n=\mathrm{O}\left(\zeta_n\right)$ with $\xi_n=\frac{n}{2}u_3^*\in\mathfrak{g}^*$, with stabilizer of ξ_n for coadjoint action the torus $K=\left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}, \theta\in\mathbb{R}\right\}$ with Lie algebra $\mathbb{R}u_3$. $\mathrm{O}_n=\mathrm{O}\left(\zeta_n\right)$ is associated with a holomorphic discrete series representation π_n of G by the KKS (Kirillov-Kostant-Souriau) method of orbits.

$$J: D\to \mathrm{O}_n$$

$$z\mapsto J(z)=\frac{n}{2}\left(\frac{z+z^*}{\left(1-|z|^2\right)}u_1^*+\frac{z-z^*}{i\left(1-|z|^2\right)}u_2^*+\frac{1+|z|^2}{\left(1-|z|^2\right)}u_3^*\right) \tag{34}$$

Group G act on D by homography $g.z=\begin{pmatrix} a & b \\ b^* & a^*\end{pmatrix}.z=\frac{az+b}{b^*z+a^*}$. ***This action corresponds with coadjoint action of*** G ***on*** O_n. The Kirillov-Kostant-Souriau 2-form of O_n is given by:

$$\Omega_n\left(\zeta\right)\left(X\left(\zeta\right),Y\left(\zeta\right)\right)=\left\langle\zeta,\left[X,Y\right]\right\rangle \ , X,Y\in\mathfrak{g} \text{ and } \zeta\in\mathrm{O}_n \tag{35}$$

and is associated in the frame by J with: $\omega_n=\frac{in}{\left(1-|z|^2\right)^2}dz\wedge dz^*$ (36)

with the corresponding Poisson Bracket: $\left\{f,g\right\}=i\left(1-|z|^2\right)^2\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z^*}-\frac{\partial f}{\partial z^*}\frac{\partial g}{\partial z}\right)$ (37)

It has been also observed that there are 3 basic observables generating the $SU(1,1)$ symmetry on classical level:

$$\begin{cases} D \to \mathbb{R} \\ z \mapsto k_3(z) = \dfrac{1+|z|^2}{1-|z|^2} \end{cases}, \begin{cases} D \to \mathbb{R} \\ z \mapsto k_1(z) = \dfrac{1}{i}\dfrac{z-z^*}{1-|z|^2} \end{cases}, \begin{cases} D \to \mathbb{R} \\ z \mapsto k_2(z) = \dfrac{z+z^*}{1-|z|^2} \end{cases} \quad (38)$$

With the Poisson commutation rule: $\{k_3,k_1\} = k_2, \{k_3,k_2\} = -k_1, \{k_1,k_2\} = -k_3$ (39)

(k_1,k_2,k_3) vector points to the upper sheet of the two-sheeted hyperboloid in $\mathbb{R}^3$ given by $k_3^2 - k_1^2 - k_2^2 = 1$, whose the stereographic projection onto the open unit disk is:

$$\begin{cases} (k_1,k_2,k_3) \in \mathrm{H}^+ \to D \\ z = \dfrac{k_2 + ik_1}{1+k_3} = \sqrt{\dfrac{k_3 - 1}{k_3 + 1}} e^{i \arg z} \end{cases} \quad (40)$$

Under the action of $g \in G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} / |a|^2 - |b|^2 = 1,\ a,b \in \mathbb{C} \right\}$:

$\begin{pmatrix} k_- & k_3 \\ k_3 & k_+ \end{pmatrix} = \begin{pmatrix} k_2 + ik_1 & k_3 \\ k_3 & k_2 - ik_1 \end{pmatrix} = \dfrac{1}{1-|z|^2} \begin{pmatrix} 2z & 1+|z|^2 \\ 1+|z|^2 & 2z^* \end{pmatrix}$ is transform in:

$$\begin{pmatrix} k_-' & k_3' \\ k_3' & k_+' \end{pmatrix} = \begin{pmatrix} k_-\left(g^{-1}.z\right) & k_3\left(g^{-1}.z\right) \\ k_3\left(g^{-1}.z\right) & k_+\left(g^{-1}.z\right) \end{pmatrix} = g^{-1} \begin{pmatrix} k_- & k_3 \\ k_3 & k_+ \end{pmatrix} \left(g^{-1}\right)^t \quad (41)$$

This transform can be viewed as the co-adjoint action of $SU(1,1)$ ***on the coadjoint orbit identified with*** $k_3^2 - k_1^2 - k_2^2 = 1$.

4 Covariant Gibbs Density by Souriau Thermodynamics

Representation theory studies abstract algebraic structures by representing their elements as linear transformations of vector spaces, and algebraic objects (Lie groups, Lie algebras) by describing its elements by matrices and the algebraic operations in terms of matrix addition and matrix multiplication, reducing problems of abstract algebra to problems in linear algebra. Representation theory generalizes Fourier analysis via harmonic analysis. The modern development of Fourier analysis during XXth century has explored the generalization of Fourier and Fourier-Plancherel formula for non-commutative harmonic analysis, applied to locally compact non-Abelian groups. This has been solved by geometric approaches based on “orbits methods” (Fourier-Plancherel formula for *G* is given by coadjoint representation of *G* in dual vector space of its Lie algebra) with many contributors (Dixmier, Kirillov, Bernat, Arnold, Berezin, Kostant, Souriau, Duflo, Guichardet, Torasso, Vergne, Paradan, etc.).

For classical commutative harmonic analysis, we consider the following groups :

$G = \mathrm{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ for Fourier series, $G = \mathbb{R}^n$ for Fourier Transform

G group character (linked to e^{ikx}): $\chi : G \to U$ with $U = \left\{ z \in \mathbb{C} / |z| = 1 \right\}$

$\hat{G} = \left\{ \chi / \chi_1 . \chi_2(g) = \chi_1(g)\chi_2(g) \right\}$ and Fourier transform is given by:

$$\begin{matrix} \varphi : G \to \mathbb{C} \\ g \mapsto \varphi(g) = \int_{\hat{G}} \hat{\varphi}(\chi)\chi(g)^{-1} d\chi \end{matrix} \quad \text{and} \quad \begin{matrix} \hat{\varphi} : \hat{G} \to \mathbb{C} \\ \chi \mapsto \hat{\varphi}(\chi) = \int_G \varphi(g)\chi(g)dg \end{matrix} \tag{44}$$

For non-commutative harmonic analysis, Group unitary irreductible representation is $\mathrm{U} : G \to U(\mathrm{H})$ with H Hilbert space and character by $\chi_{\mathrm{U}}(g) = tr\mathrm{U}_g$. Fourier transform for non-commutative group is $\mathrm{U}_\varphi = \int_G \varphi(g)\mathrm{U}_g dg$ with character $\chi_{\mathrm{U}}(g) = tr\mathrm{U}_\varphi$.

If we describe group element with exponential map $\mathrm{U}_\psi = \int_{\mathfrak{g}} \psi(X)\mathrm{U}_{\exp(X)} dX$, we have:

$$\mathrm{trU}_\psi = \dim \tau . \mu_{G.f}\left(\psi . \hat{j}^{-1} \right) \quad \text{with} \left\{ \begin{array}{l} \mu_{G.f} : \text{Liouville meas. on } \mathrm{O} = G.f,\ f \in \mathfrak{g}^* \\ \mu_{G.f}\left(\psi . \hat{j}^{-1} \right) : \text{Integral of } \psi . \hat{j}^{-1} \text{ wrt } \mu_{G.f} \end{array} \right. \tag{45}$$
$$\psi . \hat{j}^{-1} : \mathfrak{g} \to \mathfrak{g}^*, \text{ Four. Transf.}$$

where $j(X) = \left(\det s(ad_X) \right)^{1/2}$ with $s(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{x}{2} \right)^{2n} = sh\left(\frac{x}{2} \right) \Big/ \left(\frac{x}{2} \right)$ (46)

Kirillov Character formula is: $\chi_U\left(\exp(X) \right) = \mathrm{trU}_{\exp(X)} = j(X)^{-1} \int_{\mathrm{O}} e^{i\langle f, X \rangle} d\mu_{\mathrm{O}}(f)$ (47)

$$\int_{\mathrm{O}} e^{i\langle f, X \rangle} d\mu_{\mathrm{O}}(f) = j(X)\mathrm{trU}_{\exp(X)} \text{ with } j(X) = \left(\det\left(\frac{e^{ad_X/2} - e^{-ad_X/2}}{ad_X/2} \right) \right)^{1/2} \tag{48}$$

We will use Kirillov representation theory and his character formula [10-19] to compute Souriau covariant Gibbs density in the unit Poincaré disk. For any Lie group G, a coadjoint orbit $\mathrm{O} \subset \mathfrak{g}^*$ has a canonical symplectic form ω_{O} given by KKS 2-form. As seen, if G is finite dimensional, the corresponding volume element defines a G-invariant measure supported on O, which can be interpreted as a tempered distribution. The Fourier transform (where *d* is the half of the dimension of the orbit O) :

$$\Im(x) = \int_{\mathrm{O} \subset \mathfrak{g}^*} e^{-i\langle x, \lambda \rangle} \frac{1}{d!} d\omega_{O^d} \quad \text{with } \lambda \in \mathfrak{g}^* \text{ and } x \in \mathfrak{g} \tag{49}$$

is $\mathrm{Ad}\,G$-invariant. When $\mathrm{O} \subset \mathfrak{g}^*$ is an integral coadjoint orbit, Kirillov formula, given previously, expresses Fourier transform $\Im(x)$ by Kirillov character χ_{O}:

$$\Im(x) = j(x)\chi_{\mathrm{O}}\left(e^x \right) \quad \text{where } j(x) = \det{}^{1/2}\left(\frac{\sinh\left(ad\left(x/2 \right) \right)}{ad\left(x/2 \right)} \right) \tag{50}$$

χ_{O} is, as defined previously, the "*Kirillov character*" of a unitary representation associated to the orbit. We will consider the universal covering of $PSU(1,1)$, the Lie algebra is:

$$\mathfrak{g}^* = \mathfrak{su}(1,1)^* = \left\{ \begin{pmatrix} iE & p^* \\ p & -iE \end{pmatrix} / E \in \mathbb{R}, p \in \mathbb{C} \right\} \tag{51}$$

As observed in [8], the Ad-invariant form $m^2 = E^2 - |p|^2$ allows to identify the following operator Ad and Ad^*, m could be considered analogously as rest mass, E as energy, and $p = p_1 + ip_2$ as the momentum vector. The coadjoint orbits are the rest mass shells. Let $D = \{w \in \mathbb{C} / |w| < 1\}$ Poincaré unit disk, for any $m > 0$, there is a corresponding action of the universal covering of $PSU(1,1)$ on $\kappa^{m/2}$ (with κ the holomorphic cotangent bundle of unit disk), with the invariant symplectic form

$$\omega = curv(\kappa) = -i\partial\partial^* \log |dw|^2 = 2i \frac{dw \wedge dw^*}{\left(1-|w|^2\right)^2} \tag{52}$$

The moment map is an equivariant isomorphism (O_m^+ coadjoint orbit for $m^2 > 0$ and $E > 0$): $J : w \in \left(D, curv\left(\kappa^{m/2}\right)\right) \mapsto (p, E) = \frac{m}{\left(1-|w|^2\right)}\left(2iw, 1+|w|^2\right) \in \mathrm{O}_m^+$ (53)

In case $m > 1$, the Kirillov character formula is given by:

$$\chi_m\left(\exp\left(\begin{pmatrix} x & . \\ . & -x \end{pmatrix}\right)\right) = j(x)^{-1} \int_{\mathrm{O}_{m-1}^+} e^{-i\left\langle \begin{pmatrix} x & . \\ . & -x \end{pmatrix}, \begin{pmatrix} iE & p^* \\ p & -iE \end{pmatrix} \right\rangle} \omega_{\mathrm{O}_{m-1}^+} \tag{54}$$

where $j(x) = \det^{1/2}\left[\sinh\left(ad\begin{pmatrix} x/2 & \\ & -x/2 \end{pmatrix}\right) / ad\begin{pmatrix} x/2 & \\ & -x/2 \end{pmatrix}\right] = \frac{\sinh(x)}{x}$ (55)

which reduces to : $\frac{e^{mx}}{1-e^{2x}} j(x) = \int_D e^{(m-1)x\frac{1+|w|^2}{1-|w|^2}} \frac{1}{\left(1-|w|^2\right)^2} dw \wedge dw^*$ (56)

Finally, the Souriau-Gibbs density is given by:

$$p_{Gibbs}(w) = \frac{e^{-\left\langle \begin{pmatrix} ix & -\eta \\ -\eta^* & -ix \end{pmatrix}, \begin{pmatrix} im\frac{1+|w|^2}{1-|w|^2} & 2m\frac{w}{1-|w|^2} \\ 2m\frac{w}{1-|w|^2} & -im\frac{1+|w|^2}{1-|w|^2} \end{pmatrix} \right\rangle}}{j(x)\chi_m\left(e^{\begin{pmatrix} x & i\eta \\ i\eta^* & -x \end{pmatrix}}\right)} = \frac{e^{2m\left(x\frac{1+|w|^2}{1-|w|^2} + \frac{w(\eta+\eta^*)}{1-|w|^2}\right)}}{j(x)\chi_m\left(e^{\begin{pmatrix} x & i\eta \\ i\eta^* & -x \end{pmatrix}}\right)} \tag{57}$$

5 Extension from Poincaré to Siegel Homogeneous Domains

V. Bargmann has proposed the covering of the general symplectic group $Sp(2N, \mathbb{R})$:

$$Sp\left(2N,\mathbb{R}\right)=\left\{g=\begin{pmatrix}A & B\\ C & D\end{pmatrix} / \, gJ_{2N}g^{T}=J_{2N}, J_{2N}^{T}=-J_{2N}, J_{2N}=\begin{pmatrix}0 & I_N\\ -I_N & 0\end{pmatrix}\right\} \quad (58)$$

$$AB^T=BA^T, AC^T=CA^T, BD^T=DB^T, CD^T=DC^T, AD^T-BC^T=I_N \quad (59)$$

Bargmann has observed that although $Sp\left(2N,\mathbb{R}\right)$ is not isomorphic to any pseudo-unitary group, its inclusion in $U\left(N,N\right)$ will display the connectivity properties through its unitary $U\left(N\right)$ maximal compact subgroup, generalizing the role of $U\left(1\right)=SO(2)$ in $Sp\left(2,\mathbb{R}\right)$: $W_N=W\otimes I_N$, $2N\times 2N$ matrix

where $W=W_1=\frac{1}{\sqrt{2}}\begin{pmatrix}\omega_{\pi/4}^{-1} & \omega_{\pi/4}^{-1}\\ -\omega_{\pi/4} & \omega_{\pi/4}\end{pmatrix}$ with $\omega=e^{i\pi/4}=\frac{1}{\sqrt{2}}\left(1+i\right)$ (60)

$$u\left(g\right)=W_N^{-1}gW_N=\frac{1}{2}\begin{pmatrix}\left[A+D\right]-i\left[B-C\right] & \left[A-D\right]+i\left[B+C\right]\\ \left[A-D\right]-i\left[B+C\right] & \left[A+D\right]+i\left[B-C\right]\end{pmatrix}=\begin{pmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{pmatrix} \quad (61)$$

with $\alpha\alpha^+-\beta\beta^+=I_N, \alpha^+\alpha-\beta^T\beta^*=I_N$ and $\alpha\beta^T-\beta\alpha^T=0, \alpha^T\beta^*-\beta^+\alpha=0$ (62)

The symplecticity property of g becomes:

$$uM_{2N}u^+=M_{2N}, M_{2N}=iW_N^{-1}J_{2N}W_N=\begin{pmatrix}I_N & 0\\ 0 & -I_N\end{pmatrix} \quad (63)$$

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=g(u)=W_NuW_N^{-1}=\begin{pmatrix}\mathrm{Re}\left(\alpha+\beta\right) & -\mathrm{Im}\left(\alpha-\beta\right)\\ \mathrm{Im}\left(\alpha+\beta\right) & \mathrm{Re}\left(\alpha-\beta\right)\end{pmatrix} \quad (64)$$

References

1. Bargmann, V. : Irreducible unitary representations of the Lorentz group. Ann. Math. 48, pp.588-640, (1947).
2. Souriau, J.-M. : Mécanique statistique, groupes de Lie et cosmologie, Colloques int. du CNRS numéro 237. Aix-en-Provence, France, 24–28, pp. 59–113, (1974)
3. Souriau, J.-M. : Structure des systèmes dynamiques, Dunod, (1969).
4. Kirillov, A.A. : Elements of the theory of representations, Springer-Verlag, Berlin, (1976).
5. Marle, C.-M. : From Tools in Symplectic and Poisson Geometry to J.-M. Souriau's Theories of Statistical Mechanics and Thermodynamics. Entropy, 18, 370, (2016).
6. Barbaresco, F. : Higher Order Geometric Theory of Information and Heat Based on Poly-Symplectic Geometry of Souriau Lie Groups Thermodynamics and Their Contextures: The Bedrock for Lie Group Machine Learning. Entropy, 20, 840, (2018).
7. Cishahayo C., de Bièvre S. : On the contraction of the discrete series of SU(1;1), Annales de l'institut Fourier, tome 43, no 2, p. 551-567, (1993).
8. Cahen B. : Contraction de SU(1,1) vers le groupe de Heisenberg, Travaux mathématiques, Fascicule XV, pp.19-43, (2004).
9. Cahen, M., Gutt, S. and Rawnsley, J. : Quantization on Kähler manifolds I, Geometric interpretation of Berezin quantization, J. Geom. Phys. 7,45-62, (1990).
10. Dai,J. : Conjugacy classes, characters and coadjoint orbits of DiffS^1, PhD dissertation, The University of Arizona, Tucson, AZ, 85721, USA, (2000).

11. Dai J., Pickrell D. : The orbit method and the Virasoro extension of Diff+(S1): I. Orbital integrals, Journal of Geometry and Physics, n°44, pp.623-653, (2003).
12. Knapp, A. : Representation Theory of Semisimple Groups: An Overview based on Examples, Princeton University press, (1986).
13. Frenkel, I. : Orbital theory for affine Lie algebras, Invent. Math. 77, pp. 301–354, (1984).
14. Libine, M. : Introduction to Representations of Real Semisimple Lie Groups, arXiv:1212.2578v2, (2014).
15. Guichardet, A. : La methode des orbites: historiques, principes, résultats. Leçons de mathématiques d'aujourd'hui, Vol.4, Cassini, pp. 33-59, (2010).
16. Vergne, M. : Representations of Lie groups and the orbit method, Actes Coll. Bryn Mawr, p.59-101, Springer, (1983).
17. Duflo, M. ; Heckman, G. ; Vergne, M.: Projection d'orbites, formule de Kirillov et formule de Blattner, Mémoires de la SMF, Série 2, no. 15, p. 65-128, (1984).
18. Witten, E: Coadjoint orbits of the Virasoro group, Com.. Math. Phys. 114, p. 1–53, (1988).
19. Pukanszky, L. : The Plancherel formula for the universal covering group of SL(2,R), Math. Ann. 156, pp.96-143, (1964).
20. Clerc, J.L.; Orsted B.: The Maslov Index Revisited, Transformation Groups, vol. 6, n°4, pp.303-320, (2001).
21. Foth, P.; Lamb M. : The Poisson Geometry of SU(1,1), Journal of Mathematical Physics, Vol. 51, (2010).
22. Perelomov, A.M. : Coherent States for Arbitrary Lie Group, Commun. math. Phys. 26, pp. 222-236, (1972).
23. Ishi, H.: Kolodziejek, B: Characterization of the Riesz Exponential Familly on Homogeneous Cones. arXiv:1605.03896, (2018).
24. Tojo, K.; Yoshino, T. : A Method to Construct Exponential Families by Representation Theory. arXiv:1811.01394, (2018)
25. Tojo, K. and Yoshino,T.: On a method to construct exponential families by representation theory, GSI'19, SPRINGER LNCS, August (2019)
26. Pukanszky, L. : The Plancherel formula for the universal covering group of SL(2,R), Math Annalen, t.156, pp.96-143, (1964)
27. Pukanszky, L. : Leçons sur les représentations des groupes, Monographies de la Société Mathématique de France, Dunod, Paris, (1967)
28. Bernat, P. & al : Représentations des groupes de Lie, Monographie de la Société Mathématique de France, Dunod, Paris, (1972)
29. Dixmier, J. : Les algèbres enveloppantes, Gauthier-Villars, Paris, (1974)
30. Duflo, M.: Construction des représentations unitaires d'un groupe de Lie, C.I.M.E., (1980)
31. Guichardet, A.: Théorie de Mackey et méthode des orbites selon M. Duflo, Expo. Math, t.3, pp.303-346, (1985)
32. Mnemné, R. & Testard, F. : Groupes de Lie classiques, Hermann (1985)
33. Yahyai, M.:Représentations étoile du revêtement universel du groupe hyperbolique et formule de Plancherel, Thèse Université de Metz, 23 Juin (1995)
34. Rais, M. : Orbites coadjointes et représentations des groupes, cours C.I.M.P.A., (1980)
35. Rais, M. : La représentation coadjointe du groupe affine, Annales de l'Institut Fourier, Tome 28, no. 1, pp. 207-237, (1978)
36. Barbaresco, F. : Souriau Exponential Map Algorithm for Machine Learning on Matrix Lie Groups, GSI'19, SPRINGER LNCS, August (2019)
37. Barbaresco, F. : Geometric Theory of Heat from Souriau Lie Groups Thermodynamics and Koszul Hessian Geometry: Applications in Information Geometry for Exponential Families. Entropy, 18, 386, (2016)