New Generation of Statistical Radar Processing based on Geometric Science of Information

Information Geometry, Metric Spaces and Lie Groups Models of Radar Signal Manifolds

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***Abstract—* New breakthroughs have emerged in the framework of geometric Multivariate Statistical Data Processing, with many applications in Radar. We will introduce some of these basic tools from Geometric Science of Information, and more especially Radar applications based on Information Geometry and Fisher metric for Pulse Compression, CFAR, Anti-Jamming, STAP, NCTR and Tracking . We will also introduce new extensions based on Jean-Marie Souriau and Jean-Louis Koszul works, respectively on Geometric Statistical Physics and on Hessian Geometry. This paper will give references of THALES work for more than 10 years, with some parts funded by French MoD DGA/MRIS by PhDs.**

Keywords – geometric science of information; information geometry; metric spaces; Lie Groups; Fisher Metric, Statistical Radar Processing, CFAR, Anti-Jamming, STAP, NCTR, Tracking

I. Introduction

This paper is motivated by the new development of innovative tools based on differential geometry and Lie Group theory for Statistical Radar Processing. These new tools are not familiar for the Radar experts' community who classically used methods from linear algebra. The main objectives of this paper are to introduce basic references related to principles of "Information Geometry" and their uses for radar applications (with different Use-Cases): robust Doppler CFAR processing coupling Ordered statistic and high-resolution Doppler Spectrum (based on geometry of Toeplitz Hermitian Positive Definite Matrices), robust Parametrized STAP processing (based on geometry of Toeplitz-Block-Toeplitz Hermitian Positive Definite Matrices), Anti-Jamming based on Geometric Mean/Median Covariance matrix, Phase Denoising with anisotropic Diffusion and Median filter, Clutter segmentation/mapping in Metric space and tracking for fast manoeuvers detections. References to extension for non-stationary signal will be given where detection or NCTR problems are modeled as distance definition between paths on manifold. Main outcomes of this paper are better understandings of differential geometric tools available for Radar Signal Processing.

Since mid of last century, Radar Signal Processing problems have been formalized with linear functions on vector spaces and solved with classical Linear Algebra, and have used only a small part of tools that have been developed in mathematics. On the contrary in Physics, progresses have been accomplished by intensive adaptation of modern geometric theories. Differential manifolds could be naturally introduced in Radar Signal Processing by considering Information Geometry theory based on seminal paper of Rao and Fréchet, where the Fisher Matrix defines a metric in parameters space of density of probabilities. This Fisher differential metric defines a natural "information" manifold for raw radar data. These geometric tools lead to greater insight and better Radar signal algorithms performances due to their "intrinsic" properties to solve complex problems, where classically Linear Algebra uses ad-hoc projections approaches. The paper will give references to theses new geometries in the framework of Radar Signal Processing: Information Geometry for Radar statistical laws, Metric space and Cartan-Siegel Homogeneous Bounded domains Geometry for structured Radar covariance matrices and general framework of Koszul Hessian Geometry for Maximum Entropy estimation. Associate tools for these geometries will be given in references like Siegel metric for Hermitian Positive Definite matrices, Partial Iwasawa Decomposition for Toeplitz HPD matrices, Mostow/Berger fibration for Toeplitz-Block-Toeplitz Matrices, Karcher flow on manifold, p-mean barycenters on Fréchet metric spaces.

We will give generalized definition of Fisher Metric by Koszul's hessian definition proposed in pure geometry and Souriau's covariant definition proposed in Geometric Mechanics and Lie Group Thermodynamics (with notions of Geometric Temperature/Capacity and Euler-Poincaré-Souriau Equation). A general definition of Maximum Entropy Solution for Density of probabilities is linked for general case and for Hermitian Positive Definite Covariance matrices.

II. Geometric Science of Information

A. *Seminal work of Fréchet and Koszul*

Based on his IHP Lecture of Winter 1939, Maurice Fréchet wrote a seminal paper introducing what was then called the Cramer-Rao bound. This paper contains in fact much more that this important discovery. In particular, Maurice Fréchet introduced "distinguished functions", densities with estimator reaching this Cramer-Rao-Fréchet bound. He has discovered that this density depends of a function that should be solution of Clairaut's equation. The solutions "envelope of the Clairaut's equation" are equivalent to standard Legendre transform. This Fréchet-Clairaut equation can be revisited on the basis of Jean-Louis Koszul works as seminal foundation of "Information Geometry". We have shown that Fréchet discovery and Information Geometry are founded on Koszul-Vinberg Characteristic function $\psi_\Omega(x)=\int_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi,\ \forall x\in\Omega$ where Ω is a convex cone and Ω^* the dual cone with respect to Cartan-Killing inner product $\langle x,y\rangle=-B(x,\theta(y))$ invariant by automorphisms of Ω, with $B(.,.)$ the Killing form and $\theta(.)$ the Cartan involution. This characteristic function is at the cornerstone of Information Geometry, defining Koszul density by Solution of Maximum Koszul-Shannon Entropy $\Phi^*(\bar{\xi})=-\int_{\Omega^*} p_{\bar{\xi}}(\xi)\log p_{\bar{\xi}}(\xi)d\xi$:

$$\underset{p}{Max}\left[-\int_{\Omega^*} p_{\bar{\xi}}(\xi)\log p_{\bar{\xi}}(\xi)d\xi\right] \tag{1}$$

such that $\int_{\Omega^*} p_{\bar{\xi}}(\xi)d\xi=1$ and $\int_{\Omega^*}\xi.p_{\bar{\xi}}(\xi)d\xi=\bar{\xi}$

The solution is given by Gibbs density:

$$p_{\bar{\xi}}(\xi)=\frac{e^{-\left\langle \xi,\Theta^{-1}(\bar{\xi})\right\rangle}}{\int_{\Omega^*} e^{-\left\langle \xi,\Theta^{-1}(\bar{\xi})\right\rangle}d\xi} \text{ with } \bar{\xi}=\Theta(x)=\frac{\partial\Phi(x)}{\partial x} \tag{2}$$

where $\Phi(x)=-\log\int_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi$

The inversion $\Theta^{-1}(\bar{\xi})$ is given by the Legendre transform based on the property that the Koszul-Shannon Entropy is given by the Legendre transform of minus the logarithm of the characteristic function:

$$\Phi^*(x^*)=\left\langle x,x^*\right\rangle-\Phi(x) \quad \text{with } \Phi(x)=-\log\int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi \quad \forall x\in\Omega \text{ and } \forall x^*\in\Omega^* \tag{3}$$

We can observe the fundamental property that $E\left[\Phi^*(\xi)\right]=\Phi^*(E[\xi])$, $\xi\in\Omega^*$, and also as observed by Maurice Fréchet that "distinguished functions" (densities with estimator reaching the Fréchet-Darmois bound) are solutions of the ***Alexis Clairaut Equation*** introduced by Clairaut in 1734:

$$\Phi^*(x^*)=\left\langle\Theta^{-1}(x^*),x^*\right\rangle-\Phi\left[\Theta^{-1}(x^*)\right] \quad \forall x^*\in\{\Theta(x)/x\in\Omega\} \tag{4}$$

In this structure, the Fisher metric $I(x)$ makes appear naturally a ***Koszul hessian geometry***, if we observe that

$$\log p_x(\xi)=-\langle x,\xi\rangle+\Phi(x)\Rightarrow\frac{\partial^2\log p_x(\xi)}{\partial x^2}=\frac{\partial^2\Phi(x)}{\partial x^2}.$$

$$I(x)=-E_\xi\left[\frac{\partial^2\log p_x(\xi)}{\partial x^2}\right]=-\frac{\partial^2\Phi(x)}{\partial x^2}=\frac{\partial^2\log\psi_\Omega(x)}{\partial x^2} \tag{5}$$

$$I(x)=E_\xi\left[\xi^2\right]-E_\xi\left[\xi\right]^2=Var(\xi)$$

with Crouzeix relation established in 1977, $\frac{\partial^2\Phi}{\partial x^2}=\left[\frac{\partial^2\Phi^*}{\partial x^{*2}}\right]^{-1}$ giving the dual metric, in dual space, where Entropy Φ^* and (minus) logarithm of characteristic function Φ are dual potential functions.

B. General definition of Fisher Metric by J.M. Souriau

To recover the covariance of Gibbs equilibrium, Souriau has generalized this Fisher metric for Statistical Physics on Symplectic manifold, and interpreted the Fisher Metric as a Geometric Heat Capacity. Souriau has defined Gibbs canonical ensemble on Symplectic manifold M for a Lie group action on M. In classical statistical mechanics, a state is given by the solution of Liouville equation on the phase space, the partition function. As Symplectic manifolds have a completely continuous measure, invariant by diffeomorphisms, the Liouville measure λ, all statistical states will be the product of Liouville measure by the scalar function given by the generalized partition function $e^{\Phi(\beta)-\langle\beta,U(\xi)\rangle}$ defined by the energy U (defined in dual of Lie Algebra of this dynamical group) and the geometric temperature β, where Φ is a normalizing constant such the mass of probability is equal to 1, $\Phi(\beta)=-\log\int_M e^{-\langle\beta,U(\xi)\rangle}d\lambda$. Jean-Marie Souriau then generalizes the Gibbs equilibrium state to all Symplectic manifolds that have a dynamical group. To ensure that all integrals, that will be defined, could converge, ***the canonical Gibbs ensemble is the largest open proper subset (in Lie algebra) where these integrals are convergent. This canonical Gibbs ensemble is convex***. The derivative of Φ, $Q=\frac{\partial\Phi}{\partial\beta}$ (thermodynamic heat) is equal to the mean value of the energy U. The minus derivative of this generalized heat Q, $K=-\frac{\partial Q}{\partial\beta}$ is symmetric and positive (geometric heat capacity). Entropy s is then defined by Legendre transform of Φ, $s=\langle\beta,Q\rangle-\Phi$. If this approach is applied for the group of time translation, this is the classical thermodynamic theory. But ***Souriau has observed that if we apply this theory for non-commutative group (dynamic groups in physics), the symmetry has been broken***. ***Classical Gibbs equilibrium states are no longer invariant by this group***. This symmetry breaking provides new equations, discovered by Souriau.

For each temperature β, Souriau has introduced a tensor $\tilde{\Theta}_\beta$, equal to the sum of the cocycle $\tilde{\Theta}$ and the Heat coboundary (with [.,.] Lie bracket):

$$\tilde{\Theta}_\beta(Z_1,Z_2)=\tilde{\Theta}(Z_1,Z_2)+\left\langle Q,ad_{Z_1}(Z_2)\right\rangle \quad \text{with } ad_{Z_1}(Z_2)=[Z_1,Z_2] \tag{6}$$

This tensor $\tilde{\Theta}_\beta$ has the following properties:

$\tilde{\Theta}(X,Y)=\left\langle\Theta(X),Y\right\rangle$ where the map Θ is the one-cocycle of the Lie algebra $\mathfrak{g}$ with values in $\mathfrak{g}^*$, with $\Theta(X) = T_e\theta(X(e))$ where θ the one-cocycle of the Lie group G. $\tilde{\Theta}(X,Y)$ is constant on M and the map $\tilde{\Theta}(X,Y): \mathfrak{g}\times\mathfrak{g}\rightarrow\Re$ is a skew symmetric bilinear form, and is called the Symplectic Cocycle of Lie algebra $\mathfrak{g}$ associated to the momentum map J, with the following property:

$$\tilde{\Theta}(X,Y) = J_{[X,Y]} - \{J_X, J_Y\} \tag{7}$$

with $\{.,.\}$ Poisson Bracket and J the Moment Map

$$\tilde{\Theta}([X,Y],Z) + \tilde{\Theta}([Y,Z],X) + \tilde{\Theta}([Z,X],Y) = 0 \tag{8}$$

where J_X linear application from $\mathfrak{g}$ to differential function on M: $\begin{array}{l}\mathfrak{g}\rightarrow C^\infty(M,R)\\ X\rightarrow J_X\end{array}$ and the associated differentiable application J, called moment(um) map:

$$J: M\rightarrow\mathfrak{g}^* \text{ with } x\mapsto J(x) \\ \text{such that } J_X(x) = \langle J(x), X\rangle,\ X\in\mathfrak{g} \tag{9}$$

We can observe that the temperature $\beta\in Ker\,\tilde{\Theta}_\beta$, such that:

$$\tilde{\Theta}_\beta(\beta,\beta) = 0 \quad , \quad \forall\beta\in\mathfrak{g} \tag{10}$$

The following symmetric tensor g_β, defined on all values of $ad_\beta(.) = [\beta,.]$ is positive definite:

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2]) \geq 0 \tag{11}$$

where the linear map $ad_X\in gl(\mathfrak{g})$ is the adjoint representation of the Lie algebra $\mathfrak{g}$ defined by $X,Y\in\mathfrak{g}(=T_eG)\mapsto ad_X(Y) = [X,Y]$, and the co-adjoint representation of the Lie algebra $\mathfrak{g}$ the linear map $ad_X^*\in gl(\mathfrak{g}^*)$ which satisfies, for each $\xi\in\mathfrak{g}^*$ and $X,Y\in\mathfrak{g}: \langle ad_X^*(\xi), Y\rangle = \langle\xi, -ad_X(Y)\rangle$

These equations are universal, because they are not dependent of the symplectic manifold but only of the dynamical group G, the symplectic cocycle Θ, the temperature β and the heat Q. Souriau called this model "***Lie Groups Thermodynamics***".

This metric introduced by Souriau in Statistical Physics is an extension of Fisher metric. As for classical Fisher Metric (5), this metric is an hessian metric, defined as the hessian of $\log\int_M e^{-\langle\beta,U(\xi)\rangle}d\lambda$: If we differentiate the relation:

$$Q(Ad_g(\beta)) = Ad_g^*(Q) + \theta(g) \tag{12}$$

$$\frac{\partial Q}{\partial\beta}(-[Z_1,\beta],.) = \tilde{\Theta}(Z_1,[\beta,.]) + \langle Q, Ad_{.Z_1}([\beta,.])\rangle = \tilde{\Theta}_\beta(Z_1,[\beta,.])$$

$$= \tilde{\Theta}(Z_1,[\beta,Z_2]) + \langle Q, Ad_{.Z_1}([\beta,Z_2])\rangle = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2])$$

$$\Rightarrow -\frac{\partial^2\Phi}{\partial\beta^2} = -\frac{\partial Q}{\partial\beta} = g_\beta([\beta,Z_1],[\beta,Z_2]) = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2]) \tag{13}$$

We can observe that in the general definition of Souriau, the Fisher metric is then invariant with respect to the action of the group:

$$I(Ad_g(\beta)) = -\frac{\partial^2\left(\Phi - \langle\theta(g^{-1}),\beta\rangle\right)}{\partial\beta^2} = -\frac{\partial^2\Phi}{\partial\beta^2} = I(\beta) \tag{14}$$

and the Entropy "s" is also invariant:

$$s[Q(Ad_g(\beta))] = s(Q(\beta)) \tag{15}$$

The Fisher-Souriau metric is then given by:

$$I(\beta) = \frac{\partial^2\log\int_M e^{-\langle\beta,U(\xi)\rangle}d\lambda}{\partial\beta^2} = g_\beta([\beta,Z_1],[\beta,Z_2]) = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2])$$

The model of Souriau and the Fisher-Souriau Metric is illustrated in these following figures:

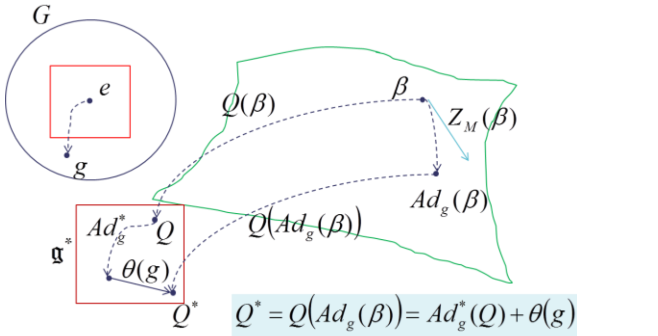

Fig. 1. Action of the group G on temperature β and heat Q.

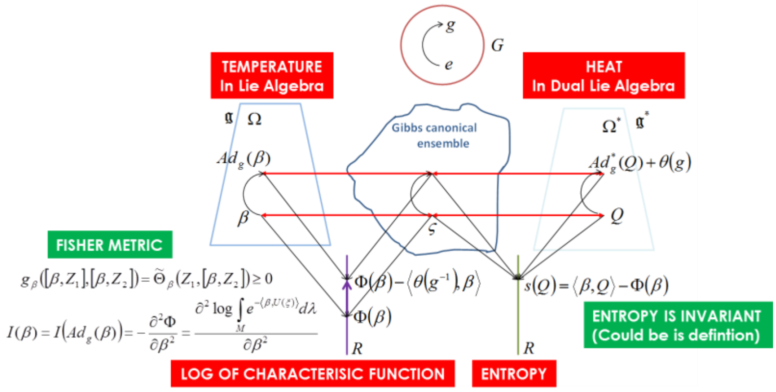

Fig. 2. Statistical Physic Model of Souriau and Fisher Metric.

We have deduced from this Souriau Model, by reduction, the Euler-Poincaré equation describing geodesic:

$$\frac{dQ}{dt} = ad_\beta^* Q \quad\text{and}\quad \begin{cases} s(Q) = \langle\beta,Q\rangle - \Phi(\beta) \\ \beta = \dfrac{\partial s(Q)}{\partial Q}\in\mathfrak{g},\ Q = \dfrac{\partial\Phi(\beta)}{\partial\beta}\in\mathfrak{g}^* \end{cases} \tag{16}$$

Back to Koszul model, we can then deduce Euler-Poincaré equation:

$$\frac{dx^*}{dt} = ad_x^* x^* \quad\text{and}\quad \begin{cases} \Phi^*(x^*) = \langle x, x^*\rangle - \Phi(x) \\ x = \dfrac{\partial\Phi^*(x^*)}{\partial x}\in\Omega,\ x^* = \dfrac{\partial\Phi(x)}{\partial x}\in\Omega^* \end{cases} \tag{17}$$

We can then deduce an Euler-Poincaré-Souriau Variational Principle for Thermodynamics. The Poincaré Theorem in Souriau Lie Group Thermodynamics is then:

The Euler-Poincaré-Souriau Variational Principle holds on $\mathfrak{g}$, for variations $\delta\beta = \dot{\eta} + [\beta, \eta]$, where $\eta(t)$ is an arbitrary path that vanishes at the endpoints, $\eta(a) = \eta(b) = 0$:

$$\delta \int_{t_0}^{t_1} \Phi(\beta(t)).dt = 0 \tag{18}$$

III. RADAR PROCESSING BASED ON GEOMETRIC SCIENCE

A. *Information Geometry for Ordered Statistics High Doppler Resolution CFAR*

We have defined a new CFAR using jointly robustness of OS (Ordered Statistics) and High Doppler Resolution (processing of covariance matrix). As there is no "total order" on covariance matrices, we use Fréchet idea of Metric Space to give a geometric definition of quantile and more especially of "median". Median of matrices will be then defined as Geodesic L1-barycenter based on metric given by Information geometry.

For Radar Processing, we will consider the parameterized density of probability $p(./\theta)$ with the metric given by the $ds^2 = K[p(./\theta), p(./\theta + d\theta)] = d\theta^+ I(\theta) d\theta$ where $I(\theta) = [g_{ij}(\theta)]$ is the Fisher Information matrix. If we model Signal by complex circular multivariate Gaussian distribution of zero mean :

$$p(X_n / R_n) = (\pi)^{-n} . |R_n|^{-1} . e^{-(X_n - m_n)^+ R_n^{-1} . (X_n - m_n)} \tag{19}$$

then for $m_n = 0$:

$$ds^2 = d\theta^+ I(\theta) d\theta = Tr\left[\left(R_n^{-1} dR_n\right)^2\right] = \left\| R_n^{-1/2} dR_n R_n^{-1/2} \right\|_F^2 \tag{20}$$

This metric has been integrated by Siegel and the distance is :

$$dist^2(R_1, R_2) = \left\| \log\left(R_1^{-1/2} . R_2 . R_1^{-1/2}\right) \right\|_F^2 = \sum_{k=1}^{n} \log^2(\lambda_k) \tag{21}$$

with $\det(R_2 - \lambda_k R_1) = 0$

This is a complete simply connected metric space of negative curvature with geodesic between R_1 and R_2 :

$$\gamma(t) = R_1^{1/2} e^{t.\log\left(R_1^{-1/2} R_2 R_1^{-1/2}\right)} R_1^{1/2} = R_1^{1/2} \left(R_1^{-1/2} R_2 R_1^{-1/2}\right)^t R_1^{1/2} \tag{22}$$

with $0 \le t \le 1$

For the robust metric and distance requested to define geodesic median L1-barycenter, we propose to use Fisher metric. As the signal is assumed to be stationary in each burst of radar cells, we can apply the Trench theorem proving that THPD (Toeplitz Hermitian Positive Definite) Covariance matrix could be parameterized by Complex Auto-Regressive (CAR) model. All THPD matrices are diffeomorphic to $(P_0, \mu_1, \ldots, \mu_n) \in R^+ x D^n$ (P_0 is a real "scale" parameter, μ_k are called reflection/Verblunsky coefficients of CAR model in D the complex unit Poincare disk, and are "shape" parameters). This Trench theorem is based on the Block Structure of THPD matrices given by:

$$R_n^{-1} = \begin{bmatrix} \alpha_{n-1} & \alpha_{n-1} A_{n-1}^+ \\ \alpha_{n-1} A_{n-1} & R_{n-1}^{-1} + \alpha_{n-1} A_{n-1} A_{n-1}^+ \end{bmatrix} \tag{23}$$

with $V^+ = (V^*)^T$, $P_0 = \alpha_0^{-1}$, $\alpha_n^{-1} = \left[1 - |\mu_n|^2\right] \alpha_{n-1}^{-1}$

$$A_n = \begin{bmatrix} A_{n-1} \\ 0 \end{bmatrix} + \mu_n \begin{bmatrix} A_{n-1}^{(-)} \\ 1 \end{bmatrix} \text{ with } V^{(-)} = J.V^*$$

where $J = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \therefore & \therefore & 0 \\ 0 & 1 & \therefore & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$

This block structure also provides iteratively André-Louis Cholesky decomposition of R_n^{-1}:

$$\Omega_n = (\alpha_n R_n)^{-1} = \Omega_n^{1/2} . \Omega_n^{1/2+} \quad , \quad \Omega_n^{1/2} = \sqrt{1 - |\mu_n|^2} \begin{bmatrix} 1 & 0_{n-1}^+ \\ A_{n-1} & \Omega_{n-1}^{1/2} \end{bmatrix} \tag{24}$$

Complex autoregressive parameters $A_n = \left[a_1^{(n)} \quad \cdots \quad a_n^{(n)}\right]^T$ and reflections coefficients $\{\mu_i\}_{i=1}^{N-1}$ are computed by ***Regularized Burg algorithm*** from pulses $\{z(k)\}_{k=1}^{N}$ of each radar burst:

$$f_0(k) = b_0(k) = z(k), \ k = 1, \ldots, N \ , \ \overline{P}_0 = \frac{1}{N} . \sum_{k=1}^{N} |z(k)|^2 \text{ and } a_0^{(0)} = 1$$

For $n = 1$ *to* $N-1$

$$\overline{\mu}_n = -\frac{\frac{2}{N-n} \sum_{k=n+1}^{N} f_{n-1}(k) . b_{n-1}^*(k-1) + 2 . \sum_{k=1}^{n-1} \beta_k^{(n)} . a_k^{(n-1)} . a_{n-k}^{(n-1)}}{\frac{1}{N-n} \sum_{k=n+1}^{N} |f_{n-1}(k)|^2 + |b_{n-1}(k-1)|^2 + 2 . \sum_{k=0}^{n-1} \beta_k^{(n)} . \left|a_k^{(n-1)}\right|^2}$$

with $\beta_k^{(n)} = \gamma (2\pi)^2 (k-n)^2$

For $k = 1$ *to* $n-1$

$$\begin{cases} a_0^{(n)} = 1 \\ a_k^{(n)} = a_k^{(n-1)} + \overline{\mu}_n . a_{n-k}^{(n-1)*} \\ a_n^{(n)} = \overline{\mu}_n \end{cases} \quad , \quad \begin{cases} f_n(k) = f_{n-1}(k) + \overline{\mu}_n . b_{n-1}(k-1) \\ b_n(k) = b_{n-1}(k-1) + \overline{\mu}_n^* . f_{n-1}(k) \end{cases}$$

In the framework of Information Geometry, we can consider this covariance matrix as a parameter for a probability density of a multivariate random process of zero mean *p(./θ)*. The fisher metric *I(θ)* defines a Riemannian metric in the space of parameters:

$$ds^2 = KL[p(Z/\theta), p(Z/\theta + d\theta)] = d\theta^+ I(\theta) d\theta = \sum_{i,j} g_{ij} d\theta_i d\theta_j^*$$

$$\text{With } g_{ij} = [I(\theta)]_{i,j} = -E\left[\frac{\partial^2 \log p(Z/\theta)}{\partial\theta_i \partial\theta_j^*}\right] \tag{25}$$

For Exponential families, the Entropy is given by $S(\bar{\theta}) = \langle \theta, \bar{\theta} \rangle - \Phi(\theta)$ with $\log p_\theta(\xi) = -\langle \theta, \xi \rangle + \Phi(\theta)$,

$$\bar{\theta} = E(\theta) \ , \ [I(\theta)]_{i,j} = -\frac{\partial^2 \Phi(\theta)}{\partial \theta_i \partial \theta_j^*} \text{ and } \left[\frac{\partial^2 S(\bar{\theta})}{\partial \bar{\theta}_i \partial \bar{\theta}_j^*}\right] = \left[\frac{\partial^2 \Phi(\theta)}{\partial \theta_i \partial \theta_j^*}\right]^{-1}$$

We can then define a metric in dual space of $\bar{\theta} = E(\theta)$:

$$ds^2_{dual} = \sum_{i,j} g_{ij}^{dual} d\bar{\theta}_i d\bar{\theta}_j^* \text{ with } g_{ij}^{dual} = \frac{\partial^2 S(\bar{\theta})}{\partial \bar{\theta}_i \partial \bar{\theta}_j^*} \tag{26}$$

For Multivariate Gaussian Process of zero mean, Entropy is $S(\bar{R}_n) = \log\left(\det \bar{R}_n^{-1}\right) - \log(\pi.e)$, developed by use of (23):

$$S(\bar{R}_n) = -\sum_{k=1}^{n-1} (n-k) \log\left[1 - |\bar{\mu}_k|^2\right] - n \log\left[\pi.e.\bar{P}_0\right] \tag{27}$$

If we use the canonical vector of parameters:

$$\theta^{(n)} = \left[\bar{P}_0 \quad \bar{\mu}_1 \quad \cdots \quad \bar{\mu}_{n-1}\right]^T = E\left[\left[P_0 \quad \mu_1 \quad \ldots \quad \mu_{n-1}\right]^T\right] \tag{28}$$

The dual metric of Information Geometry is finally given by:

$$ds^2_{dual} = \sum_{i,j} g_{ij}^{dual} d\bar{\theta}_i^{(n)} d\bar{\theta}_j^{(n)*} = n.\left(\frac{d\bar{P}_0}{\bar{P}_0}\right)^2 + \sum_{i=1}^{n-1} (n-i) \frac{|d\bar{\mu}_i|^2}{\left(1 - |\bar{\mu}_i|^2\right)}$$

The robust "Information Geometry" distance can be computed by integration in product space $R^+ x D^n$:

$$d^2\left[\left(\bar{P}_{0,1}, \{\bar{\mu}_{i,1}\}_{i=1}^{N-1}\right), \left(\bar{P}_{0,2}, \{\bar{\mu}_{i,2}\}_{i=1}^{N-1}\right)\right] = N \log^2\left(\frac{\bar{P}_{0,2}}{\bar{P}_{0,1}}\right) + \sum_{i=1}^{N-1} (N-i) \left(\frac{1}{2} \log\left(\frac{1+\delta_i}{1-\delta_i}\right)\right)^2$$

$$\text{with } \delta_i = \left|\frac{\bar{\mu}_{i,1} - \bar{\mu}_{i,2}}{1 - \bar{\mu}_{i,1} \bar{\mu}_{i,2}^*}\right| \tag{29}$$

The L_p-barycenter is computed by Karcher Flow to minimize:

$$\left(P_{0,barycenter}, \{\mu_{i,barycenter}\}_{i=1}^{N-1}\right) = \underset{P_{0,median}, \{\mu_{i,median}\}_{i=1}^{N-1}}{ArgMin} \sum_{k=1}^{M} d^p\left[\left(P_{0,barycenter}, \{\mu_{i,barycenter}\}_{i=1}^{N-1}\right), \left(\bar{P}_{0,k}, \{\bar{\mu}_{i,k}\}_{i=1}^{N-1}\right)\right]$$

This approach has been extended for the estimation of the scatter matrix of a scale mixture of Gaussian stationary autoregressive vectors, that is equivalent to the estimation of a structured scatter matrix of a Spherically Invariant Random Vector (SIRV) whose structure comes from an autoregressive modelization. We have proposed to adapt the approach by changing the energy functional minimized in the Burg algorithm. We minimize the new enrgy functional that depends on forward f and backward b prediction error:

$$U^{(m+1)} = \sum_{n=m+2}^{d} \frac{|f_{m+1}(n)|^2 + |b_{m+1}(n)|^2}{|f_m(n)|^2 + |b_m(n-1)|^2} \tag{30}$$

The minimum of this empirical version of the energy is then :

$$\bar{\mu}_{m+1} = -\frac{2}{N(d-m-1)} \sum_{i=1}^{N} \sum_{n=m+2}^{d} \frac{b_{i,m}^*(n-1) f_{i,m}(n)}{|f_{i,m}(n)|^2 + |b_{i,m}(n-1)|^2} \tag{31}$$

As this estimator is not consistent, the bias is corrected:

$$\bar{\mu}_m^{(u)} \to B_1^{-1}\left(|\bar{\mu}_m|\right) \frac{\bar{\mu}_m}{|\bar{\mu}_m|} \text{ with } B_1(x) = \frac{1-x^2}{x} \left(\frac{\log\left(\frac{1-x}{1+x}\right)}{2x} + \frac{1}{1-x^2}\right) \tag{32}$$

We have then proved that it is useful to take into account the non-Gaussianity of the clutter for sub-exponential amplitude distributions, that considering medians instead of means in the Burg estimators furnishes a robustness for medium contamination (10% to 30% outlier samples), that 2-step procedures (consisting in a selection of secondary data) need a first estimation of the scatter matrix of the clutter that is robust enough in order to be efficient (in that case, high contamination close to 50% can be considered). We illustrate performances of this algorithm, called Poincaré Median-Burg, or 2-step Poincaré-Median if the algorithm is used with a 2 step for data cells selection. Following figures illustrate robustness performances in case of abrupt Clutter Doppler variation, and in case of outliers in learning window.

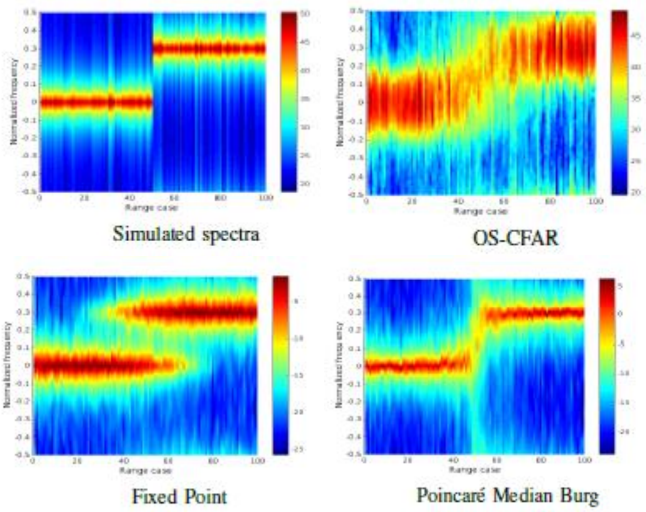

Fig. 3. Doppler/Range Spectrum in case of clutter discontinuity.

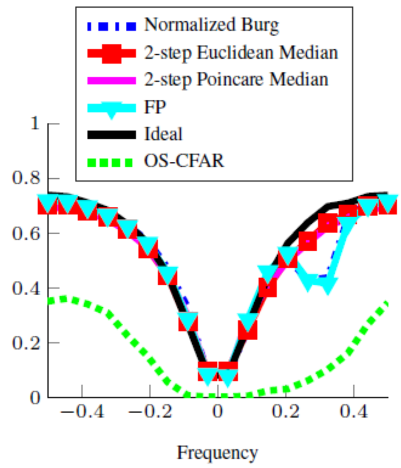

Fig. 4. Probability of detection versus normalized frequency (10 contaminating range cells with outliers at normalized frequency 0.3)

B. *Information Geometry for Ordered Statistics STAP*

We can extend previous approach for STAP by considering (Toeplitz-)Block-Toeplitz matrices. For this extension, Information metric will be introduced as previously as a Kähler potential defined by Hessian of multi-channel entropy $\tilde{\Phi}(R_{p,n+1})$:

$$\tilde{\Phi}(R_{p,n}) = \log(\det R_{p,n}^{-1}) + cte = -Tr(\log R_{p,n}) + cste \Rightarrow g_{i\bar{j}} = Hess[\phi(R_{p,n})]$$

Using partitioned matrix structure of Toeplitz-Block-Toeplitz matrix $R_{p,n+1}$, recursively parametrized by Burg-Like reflection coefficients matrix $\{A_k^k\}_{k=1}^{n-1}$ with $A_k^k \in SD_n$, we can give Multivariate entropy, matrix extension of previous Entropy :

$$\tilde{\Phi}(R_{p,n}) = -\sum_{k=1}^{n-1}(n-k).\log\det\left[1 - A_k^k A_k^{k+}\right] - n.\log\left[\pi.e.\det R_0\right] \quad (33)$$

Paul Malliavin has proved that this form is a Kähler Potential of an invariant Kähler metric (Information Geometry metric in our case) that is given by matrix extension of previous metric (29) studied for CFAR:

$$ds^2 = nTr\left[(R_0^{-1}dR_0)^2\right] + \sum_{k=1}^{n-1}(n-k)Tr\left[\left(I_n - A_k^k A_k^{k+}\right)^{-1} dA_k^k \left(I_n - A_k^{k+} A_k^k\right)^{-1} dA_k^{k+}\right]$$

As we have defined a metric space, we can extend Karcher/Frechet flow in Unit Siegel Disk to compute the Median of N Toeplitz-Block-Toeplitz Hermitian Positive Definite matrices. These matrices are parameterized by Burg-Like generalized Reflection matrices $\{A_k^k\}_{k=1}^{n-1}$ with $A_k^k \in SD_n$ and Karcher/Frechet Flow in Siegel Disk will be solved by analogy of our scheme used in Poincaré unit Disk, by mean of Mostow Decomposition Theorem : every matrix M of $GL(n,C)$ can be decomposed in $M = Ue^{iA}e^{S}$ where S is symmetric:

$$S = \frac{1}{2}\log\left(P^{1/2}\left(P^{-1/2}P^{*}P^{-1/2}\right)^{1/2}P^{1/2}\right) \text{ with } P = M^{+}M \qquad (34)$$

A is antisymmetric $A = \frac{1}{2i}\log\left(e^{-S}Pe^{-S}\right)$ and finally, $U = Me^{-S}e^{-iA}$ is unitary. Median in Siegel disk could be then obtained by analogy with scheme developed for median in Poincaré's disk. Numerical scheme of this algorithm has been recently studied by Raf Vandebril and Ben Jeuris.

C. *Information Geometry for NCTR by micro-Doppler*

Radar processing, as described previously, for non-stationary signal, corresponding to fast time variation of Doppler Spectrum in one burst, is no longer optimal. This phenomenon could be observed for high speed or abrupt Doppler variations of clutter or target signal but also in case of target migration during the burst duration due to the use of high range resolution mode. We propose new Radar Doppler processing assuming that each non-stationary signal in one burst can be split into several short signals with less Doppler resolution but locally stationary, represented by time series of Toeplitz covariance matrices. In Information Geometry (IG) framework, these time series could be defined as a geodesic path (or geodesic polygon in discrete case) on covariance Toeplitz Hermitian Positive Definite matrix manifold. For this micro-Doppler analysis, we generalize the Fréchet distance between two curves in the plane to geodesic paths in abstract IG metric spaces of covariance matrix manifold. This approach could be used for robust detection of target in case of non-stationary Time-Doppler spectrum or for NCTR function based on micro-Doppler analysis and "Deep Learning".

We consider the Fréchet distance between two curves, that is defined as the minimum length of a leash required to connect a dog and its owner as they walk without backtracking along their respective curves from one endpoint to the other. The Fréchet metric takes the flow of the two curves into account; the pairs of points whose distance contributes to the Fréchet distance sweep continuously along their respective curves.

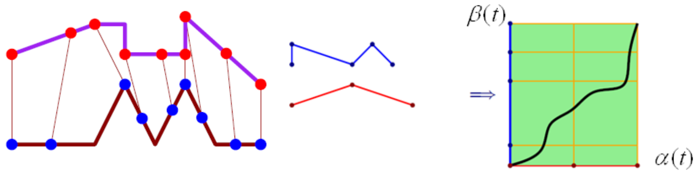

Fig. 5. Fréchet Distance between two polygonal curves (α and β indexing all matching of points)

Let P and Q be two given curves, the Fréchet distance between P and Q is defined as the infimum over all reparameterizations α and β of $[0,1]$ of the maximum over all $t \in [0,1]$ of the distance in between $P(\alpha(t))$ and $Q(\beta(t))$. In mathematical notation, the Fréchet distance $d_{Fréchet}(P,Q)$ is:

$$\begin{cases} d_{Fréchet}(P,Q) = \underset{\alpha,\beta}{Inf}\ \underset{t\in[0,1]}{Max}\{d(P(\alpha(t)),Q(\beta(t)))\} \\ \alpha \text{ and } \beta : [0,1] \rightarrow [0,1] \text{ Nondecreasing and surjective} \end{cases} \qquad (35)$$

Classically, the free-space diagram is used to compute Fréchet distance between two curves for a given distance threshold ε is a two-dimensional region in the parameter space that consist of all point pairs on the two curves at distance at most ε:

$$D_\varepsilon(P,Q) = \left\{(\alpha,\beta) \in [0,1]^2 \,/\, d_{Fréchet}(P(\alpha(t)),Q(\beta(t))) \le \varepsilon\right\} \quad (36)$$

The Fréchet distance $d_{Fréchet}(P,Q)$ is at most ε if and only if the free-space diagram $D_\varepsilon(P,Q)$ contains a path which from the lower left corner to the upper right corner which is monotone both in the horizontal and in the vertical direction. Therefore, if there is a monotone increasing curve from the lower left to the upper right corner of the diagram (corresponding to a monotone mapping), it generates a monotonic path that defines a matching between point-sets P and Q. As we need a unique parametrisation for the Frechet distance, corresponding to unique matching between points, we will consider then the line with the smallest length. To compute this line from the Free Space Diagram we use "Fast Marching" algorithm to solve the "eikonal equation" with the Free Space Diagram as background potential $\gamma^* = \arg\min_{\gamma} \int_{\gamma} g(\gamma).d\gamma$ with $g = I_{F_{d_F}}(.)$ is the indicator function of the set B and F_{d_F} is the Free Space

Diagram for the Frechet distance d_F. See Green Line in following Figure, solution of this shortest path problem.

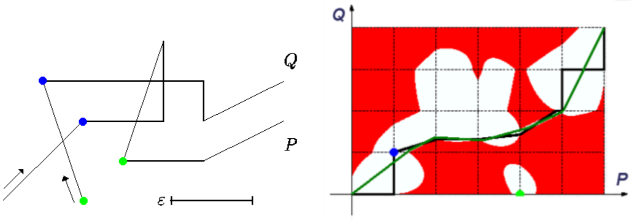

Fig. 6. Fréchet free-space diagram for 2 polygonal curves P and Q with monotonicity in both directions

We have used a second approach based on shortest path declined from Fréchet distance, by considering:

$$d_{Eikonal}(C_1,C_2)=\inf_{g:[0,l_1]\to[0,l_2]}\int d\left(C_1(s),C_2(g(s))\right)\sqrt{1+g'^2}ds \quad (37)$$

We have replaced the "Max" by "Integral" along g in the Free Space Diagram, and consider the eikonal problem to find the shortest path in Free Space Diagram considering that the metric is weighted by distance $d\left(C_1(s),C_2(g(s))\right)$ between matched points $C_1(s)$ and $C_2(g(s))$.

One drawback of the Fréchet distance is that it cannot be written as a product space; therefore we cannot compute the distance between curves as the distance between each curve component in Product space parameterization $\theta^{(n)}=\left[P_0 \quad \mu_1 \quad \cdots \quad \mu_{n-1}\right]^T$ and then sum up the contributions. It can be easily proved that:

$$d_{Fréchet}(C_1,C_2)\geq\sqrt{d_l(P_0^{(1)}(\cdot),P_0^{(2)}(\cdot))^2+\sum_{k=1}^{N-1}(N-k).d_l(\mu_k^{(1)}(\cdot),\mu_k^{(2)}(\cdot))^2}$$

Even if the distance does not separate the space as a product space, it is still useful to use this as an approximation because we will work only with curves in the Poincare disk (of reflection coefficients) to sum up the contributions. We will also neglect the reflectivity term P_0 since we are only interested in the shape of the Doppler Spectrum. Before explaining Karcher Flow to compute Median Paths, we will introduce Karcher Flow to compute barycenter in metric space. Maurice Frechet introduced in a notion of barycentre of a set of points $\{x_i, i\in[1,N]\}$ in a generic metric space E with distance d:

$$m_p=\arg\min_{x\in E}\sum_{i=1}^{N}d^p(x_i,x)=\arg\min_{x\in E} f_p(\mathrm{x}) \quad (38)$$

with $p=2$ for mean and $p=1$ for median

In particular, for a riemannian manifold of negative curvature *(M; d)* (the Poincare disk for reflection coefficients), with d the geodesic distance on M, the previous optimisation problem is strictly convex and differential if $x\in B\left(x_0,r^*\right)$ for a $x_0\in M$, where r^* is the injectivity radius of M. This has been proved by Elie Cartan. The Karcher Flow is given by:

$$m_{n+1}=\exp_{m_n}\left(-\varepsilon_n.\nabla f_1(m_n)\right) \text{ where } \nabla f_1(m_n)=-\sum_{i=1/x_i\neq m_n}^{N}\frac{\exp_{m_n}^{-1}(x_i)}{d(x_i,m_n)}$$

where

- $\exp_{m_n}(\vec{\gamma})$: is "exponential map" operator, that provides the point reached from point m_n by following geodesic path on the manifold M in the direction of the tangent vector $\vec{\gamma}$ at a geodesic distance equal to $\|\vec{\gamma}\|$.
- $\exp_{m_n}^{-1}(x_i)$: is the inverse operator, that provides tangent vector on M at point m_n of the geodesic between point m_n and point x_i.

Inspired by the Karcher flot algorithm for the mean/median of points on M, we introduce the Karcher flot for curves on M. Let C be a curve on M and let $C_{set}^K=\{C_1,...,C_K\}$ be a set of curves on the same manifold. For every pair $\{C,C_i\}$, Optimal path in Free Space Diagram ,obtained by Fréchet distance or Eikonal distance computation, will provide pair of parameterization on each curve, and matching $\gamma_{C_i}^C$ between points of C and C_i. Then if C is parametrised by its normalized arc length $C(s)$, the point $C(s)$ is matched with the point $C_i(\gamma_{C_i}^C(s))$ for $s\in[0,1]$. We define the Karcher flot for the curves as follows:

1. Select $C^{(0)}$ as an arbitrary curve on M
2. Compute Optimal Path in Free Space Diagram of $\{C^{(n)},C_i\}$ for Fréchet or Eikonal distance ➲ Optimal matching $\gamma_{C_i}^{C^{(n)}}$ and $d_{Fréchet}\left(C^{(n)},C_i\right)$ or $d_{Eikonal}\left(C^{(n)},C_i\right)$
3. Move each point of curve $C^{(n)}(s)$, according to Karcher Flow to compute barycenter of its matching point $\{C_i(\gamma_{C_i}^C(s)), i=1,...K\}$:

$$C^{(n+1)}=\left\{\exp_{C^{(n)}(s)}\left(\varepsilon_n\sum_{C_i\left(\gamma_{C_i}^{C^{(n)}}(s)\right)\neq C^{(n)}(s)}\frac{\exp_{C^{(n)}(s)}^{-1}\left(C_i\left(\gamma_{C_i}^{C^{(n)}}(s)\right)\right)}{dist\left(C_i\left(\gamma_{C_i}^{C^{(n)}}(s)\right),C^{(n)}(s)\right)}\right),s\in[0,1]\right\}$$

and re-parameterized points of $C^{(n+1)}$ according to its normalized arclength.

4. Iterate step 2 until $d_{Fréchet}\left(C^{(n)},C^{(n+1)}\right)\leq S_{threshold}$

Using short sliding window in one burst, we parameterized the non-stationary Radar signal as a time series of reflection coefficients represented as geodesic paths in Poincaré unit disk (deduced from Information Geometry metric). Fréchet distance and median barycenter of paths are used for target detection in case of non-stationarity of signal.

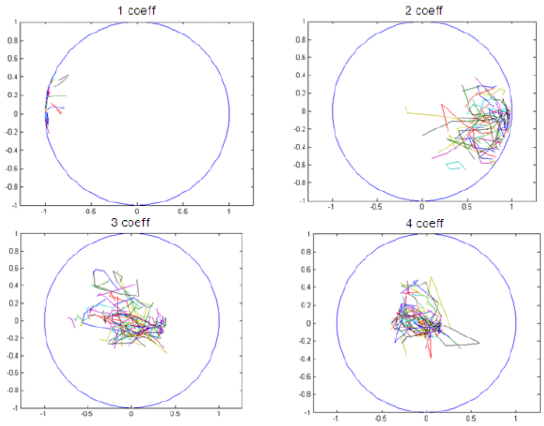

Fig. 7. Example on Ground clutter of paths in Poincaré unit disk for the 4 first reflection coefficients estimated on a sliding window in one burst

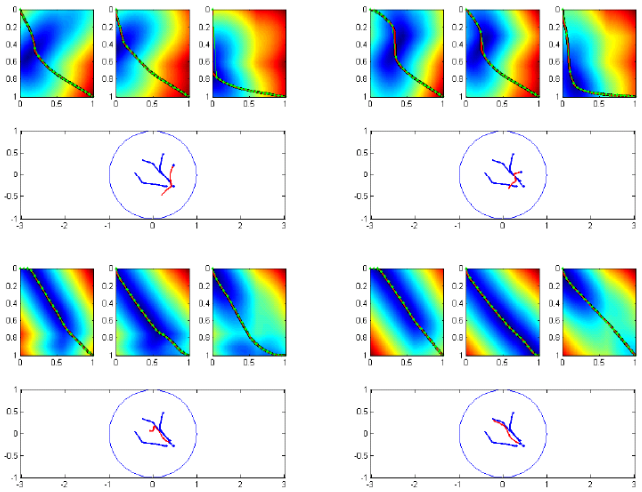

Fig. 8. Median computation but with Eikonal distance for 3 curves (in blue). At each iteration, the evolution of the curve C (in red) driven by Karcher Flow and the Free Space Diagram corresponding to matching of C with C1/C2/C3

D. *CFAR and STAP based on Kernel Density Estimation, Riemannian Mean-Shift and Statistical Depth*

To improve CFAR and STAP performance in highly non-stationary clutters, we propose new methods based on last progress in multivariate statistics and their extension in metric spaces. First, we introduce a kernel density estimation on the elements of the product $(P_0, \mu_1, \ldots, \mu_n) \in R^+ x D^n$, to estimate density for Doppler Spectrum. The specificity of the hyperbolic space enables to adapt the different density estimation methods at a reasonable cost. Let $K : R^+ \to R^+$ be a map which verifies the following properties: $\int_{R^n} K(\|x\|)dx = 1$, $\int_{R^n} x.K(\|x\|)dx = 0$, $K(x > 1) = 0$, $Sup(K(x)) = 0$

Given a point $p \in H_n$ (the hyperbolic space of dimension n; $H_2 = D$), the exponential map $\exp_p$ defines a new injective parametrization of H_n. The Lebesgue measure of the tangent space is noted Leb_p. The function $\theta_p : H_n \to R^+$ defined by:

$$\theta_p : q \to \theta_p(q) = \frac{dvol}{d\exp_p^*(Leb_p)}(q) \tag{39}$$

is the density of the Riemannian measure with respect to the image of the Lebesgue measure of $T_p H_n$ by $\exp_p$. Given K and a scaling parameter λ, the estimator of f proposed by Pelletier is defined by: $\hat{f}_k = \frac{1}{k}\sum_i \frac{1}{\lambda^n}\frac{1}{\theta_{x_i}(x)} K\left(\frac{d(x,x_i)}{\lambda}\right)$ (40)

Given $p_{ref} \in H_n$, μ_k the empirical measure and "*" the natural convolution on homogeneous spaces, let:

$$\tilde{K}(q) = \frac{1}{k\lambda^n}\frac{1}{\theta_{p_{ref}}(q)} K\left(\frac{d(p_{ref},q)}{\lambda}\right) \text{ then } \hat{f}_k = \mu_k * \tilde{K} \tag{41}$$

One still needs to obtain an explicit expression of θ_p. Given a reference point p, the point of polar coordinates (r, α) of the hyperbolic space H_n is defined as the point at distance r of p on the geodesic with initial direction $\alpha \in S^{n-1}$. Since H_n is isotropic the expression the length element in polar coordinates depends only on r. Expressed in polar coordinates the hyperbolic metric expression is: $g_{H_n} = dr^2 + \sinh(r)^2 . g_{S^{n-1}}$

The polar coordinates are a polar expression of the exponential map at p. In an adapted orthonormal basis of the tangent plane the metric takes the following form:

$$G = \begin{pmatrix} 1 & 0 \\ 0 & \sinh(r)^2 \frac{1}{r^2} I_{n-1} \end{pmatrix} \tag{42}$$

where G is the matrix of the metric and I_{n-1} is the identity matrix of size n−1. The volume volume $dvol$ is given by:

$$dvol = \sqrt{|G|}.d\exp_p^*(Leb_p) = \left(\frac{1}{r}\sinh(r)\right)^{n-1} d\exp_p^*(Leb_p) \tag{43}$$

where $r = d(p, q)$. Finally, one obtains: $\theta_p(q) = \left(\frac{1}{r}\sinh(r)\right)^{n-1}$

$$\hat{f}_k = \frac{1}{k}\sum_i \frac{1}{\lambda^n}\frac{d(x,x_i)^{n-1}}{\sinh(d(x,x_i))^{n-1}} K\left(\frac{d(x,x_i)}{\lambda}\right) \tag{43}$$

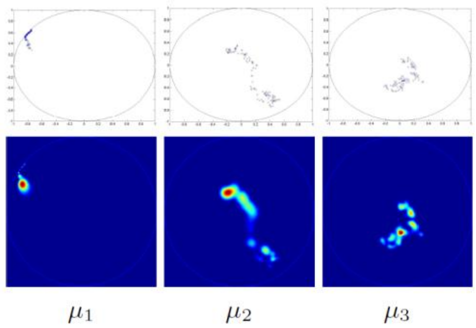

Fig. 9. Estimation of density estimation by kernel methof in Poincaré Unit Disk for the first 3 reflection coefficient (atmospheric clutter)

Based on this density estimation, we can apply segmentation in homogeneous clutter area by Riemannian Mean-Shift algorithm. The original mean shift algorithm is widely applied for nonparametric clustering of data in vector spaces. We generalize it to data points lying on Riemannian manifolds of reflection coefficients. This allows us to extend mean shift based clustering to Clutter data mapping for segmentation of area with homogeneous Doppler content. Mean shift is provided by following gradient equation where the terms lie in the tangent space, and the kernel terms K are scalars. The mean shift vector is a weighted sum of tangent vectors, and is itself a tangent vector. The mean shift iteration is: $y_{j+1} \exp_{y_j}\left(m_\lambda(y_j)\right)$ with $g(.) = -K'(.)$:

$$m_\lambda(y) = \left[\sum_{i=1}^{n} g\left(\frac{d(y,x_i)}{\lambda}\right)\right]^{-1} \sum_{i=1}^{n} g\left(\frac{d(y,x_i)}{\lambda}\right) \log_y(x_i) \qquad (44)$$

After segmentation by Riemannian Mean-Shift, we have to optimize detection by adaptive threshold. For this task, we use “Statistical depth” functions that provide from the "deepest" point a "center-outward ordering" of multidimensional data. In this sense, depth functions can measure the "extremeness" or "outlyingness" of a data point with respect to a given data set. Statistical Depth can detect targets that appear extreme relative to the rest of the observations.

E. Information Geometry for Robust Tracking by maneuver detection based on Geodesic Shooting

The development of fixed antenna radars enables to adapt the update rate of the radar. It is then necessary to decide when to generate more pulses, like during a maneuver. An efficient Maneuver detection is thus needed. Most detectors (CUSUM, GLR) measure only filtered normalized innovation. We propose a new detector based on Information geometry that compute a distance by geodesic shooting and monitor jointly bias and covariance changes for the innovation vector.

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